Name______________________________ Sec_______

1-12	/60	15	/15
13	/15	16	/ 5
14	/10	Total	/105

MATH 253 Exam 2 Fall 2009

Section 501,503 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Compute $\int_0^2 \int_{-x}^{x} 3y^2\,dy\,dx$.

a. 8 Correct Choice

b. 12

c. 16

d. 24

e. 0

$$\int_0^2 \int_{-x}^{x} 3y^2\,dy\,dx = \int_0^2 \left[y^3\right]_{y=-x}^{x} dx = \int_0^2 \left(x^3 - (-x)^3\right) dx = \int_0^2 2x^3\,dx = \left[\frac{x^4}{2}\right]_{x=0}^{2} = 8$$

2. Compute $\int_0^9 \int_{\sqrt{y}}^{3} \pi \sin(\pi x^3)\,dx\,dy$. HINT: Reverse the order of integration.

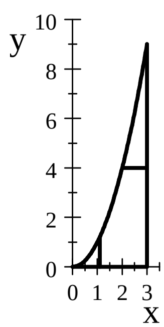
a. $\frac{1}{9}$

b. $\frac{2\pi}{9}$

c. $\frac{1}{3}$

d. $\frac{2}{3}$ Correct Choice

e. $\frac{2\pi}{3}$

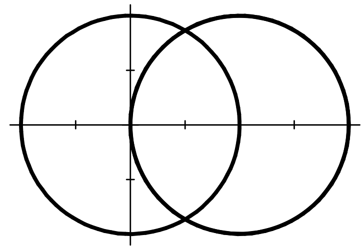


$$\int_0^9 \int_{\sqrt{y}}^{3} \pi \sin(\pi x^3)\,dx\,dy = \int_0^3 \int_0^{x^2} \pi \sin(\pi x^3)\,dy\,dx$$

$$= \int_0^3 [\pi \sin(\pi x^3) y]_{y=0}^{x^2} dx = \int_0^3 \pi x^2 \sin(\pi x^3)\,dx$$

$$= \frac{-1}{3}\cos(\pi x^3)\Big|_0^3 = \frac{-1}{3}\cos(27\pi) + \frac{1}{3}\cos(0) = \frac{2}{3}$$

3. Find the area inside the circle $r = 2\cos\theta$ but outside the circle $r = 1$.



a. $\frac{\pi}{3} - \cos\frac{\pi}{3}$

b. $\frac{2\pi}{3} + \cos\frac{2\pi}{3}$

c. $\frac{\pi}{3} + \sin\frac{2\pi}{3}$ Correct Choice

d. $\frac{2\pi}{3} - \sin\frac{\pi}{3}$

e. $\frac{2\pi}{3} + \sin\frac{\pi}{3}$

$$2\cos\theta = 1 \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = -\frac{\pi}{3}, \frac{\pi}{3}$$

$$A = \iint 1\,dA = \int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3}\left[\frac{r^2}{2}\right]_{r=1}^{2\cos\theta} d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(4\cos^2\theta - 1)\,d\theta$$
$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left(4\frac{1+\cos 2\theta}{2} - 1\right)d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(1 + 2\cos 2\theta)\,d\theta$$
$$= \frac{1}{2}\left[\theta + \sin 2\theta\right]_{\theta=-\pi/3}^{\pi/3} = \frac{1}{2}\left(\frac{\pi}{3} + \sin\frac{2\pi}{3}\right) - \frac{1}{2}\left(-\frac{\pi}{3} + \sin\frac{-2\pi}{3}\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{3} + \sin\frac{2\pi}{3} + \frac{\pi}{3} + \sin\frac{2\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3}$$

4. Compute $\int_0^2\int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx$. HINT: Switch to polar coordinates.

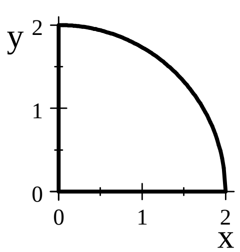
a. $\frac{\pi}{2}e^2$

b. $\frac{\pi}{2}e^4$

c. $\frac{\pi}{2}(e^2 - 1)$

d. $\frac{\pi}{2}(e^4 - 1)$

e. $\frac{\pi}{4}(e^4 - 1)$ Correct Choice



$$\int_0^2\int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^2 e^{r^2}\,r\,dr\,d\theta$$
$$= \left[\frac{\pi}{2}\right]\left[\frac{1}{2}e^{r^2}\right]_0^2 = \frac{\pi}{4}(e^4 - 1)$$

5. Compute the mass of the solid cone $\sqrt{x^2+y^2} \le z \le 4$ if the volume density is $\rho = z$.

a. 4π

b. 8π

c. 16π

d. 32π

e. 64π Correct Choice

$$M = \iiint \rho\, dV = \int_0^{2\pi}\int_0^4\int_r^4 z\,r\,dz\,dr\,d\theta = 2\pi\int_0^4 r\left[\frac{z^2}{2}\right]_{z=r}^4 dr = 2\pi\int_0^4 r\left(8-\frac{r^2}{2}\right)dr$$
$$= 2\pi\left[4r^2-\frac{r^4}{8}\right]_{r=0}^4 = 2\pi\left(4^3-\frac{4^4}{8}\right) = 2\pi 4^3\left(1-\frac{1}{2}\right) = 64\pi$$

6. Compute the center of mass of the solid cone $\sqrt{x^2+y^2} \le z \le 4$ if the volume density is $\rho = z$.

a. $\left(0, 0, \frac{5}{16}\right)$

b. $\left(0, 0, \frac{16}{5}\right)$ Correct Choice

c. $\left(0, 0, \frac{1024\pi}{5}\right)$

d. $\left(0, 0, \frac{5}{1024\pi}\right)$

e. $\left(0, 0, \frac{8}{5}\right)$

$$M_{xy} = \iiint z\rho\, dV = \int_0^{2\pi}\int_0^4\int_r^4 z^2\,r\,dz\,dr\,d\theta = 2\pi\int_0^4 r\left[\frac{z^3}{3}\right]_{z=r}^4 dr = 2\pi\int_0^4 r\left(\frac{64}{3}-\frac{r^3}{3}\right)dr$$
$$= \frac{2\pi}{3}\left[32r^2-\frac{r^5}{5}\right]_{r=0}^4 = \frac{2\pi}{3}\left(2\cdot 4^4-\frac{4^5}{5}\right) = \frac{2\pi}{3}4^4\left(2-\frac{4}{5}\right) = \frac{1024\pi}{5}$$

$\bar{z} = \frac{M_{xy}}{M} = \frac{1024\pi}{5}\frac{1}{64\pi} = \frac{16}{5}$ By symmetry $\bar{x} = \bar{y} = 0$

7. Find the average value of the function $f = \frac{1}{x^2+y^2+z^2}$ over the solid region between the two spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$. HINT: $f_{\text{ave}} = \frac{\iiint f\, dV}{\iiint 1\, dV}$

a. $\frac{3}{4}$

b. $\frac{5}{8}$

c. $\frac{3}{7}$ Correct Choice

d. 4π

e. 8π

$$\iiint f\, dV = \int_0^{\pi}\int_0^{2\pi}\int_1^2 \frac{1}{\rho^2}\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi = \left[-\cos\varphi\right]_0^{\pi}(2\pi)(2-1) = 4\pi$$

$\iiint 1\, dV =$ Volume $= \frac{4}{3}\pi 2^3 - \frac{4}{3}\pi 1^3 = \frac{28}{3}\pi$ $\quad f_{\text{ave}} = \frac{4\pi 3}{28\pi} = \frac{3}{7}$

8. Compute $\iint \cos\left(\frac{x^2}{9}+\frac{y^2}{4}\right)dx\,dy$ over the region inside the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$.

HINT: Elliptical coordinates are $x = 3t\cos\theta,\quad y = 2t\sin\theta$.

a. $6\pi\sin(1)-6\pi$

b. $6\pi\sin(1)$ Correct Choice

c. $6\pi-6\pi\cos(1)$

d. $-6\pi\cos(1)$

e. $6\pi\cos(1)$

$$J = \left|\frac{\partial(x,y)}{\partial(t,\theta)}\right| = \left\|\begin{matrix} 3\cos\theta & 2\sin\theta \\ -3t\sin\theta & 2t\cos\theta \end{matrix}\right\| = |6t\cos^2\theta + 6t\sin^2\theta| = 6t \text{ assuming } t \ge 0$$

$$\frac{x^2}{9}+\frac{y^2}{4} = \frac{(3t\cos\theta)^2}{9}+\frac{(2t\sin\theta)^2}{4} = t^2 < 1 \qquad 0 \le t \le 1$$

$$\iint \cos\left(\frac{x^2}{9}+\frac{y^2}{4}\right)dx\,dy = \int_0^{2\pi}\int_0^1 \cos(t^2)\,6t\,dt\,d\theta = 2\pi 3\sin(t^2)\big|_0^1 = 6\pi\sin(1)$$

9. Compute $\int_{(0,0)}^{(\sqrt{2},2)} x\,ds$ along the parabola $y = x^2$ parametrized as $\vec{r}(t) = (t,t^2)$.

a. $\frac{13}{6}$ Correct Choice

b. $\frac{9}{4}$

c. $\frac{27}{4}$

d. $\frac{1}{12}(17^{3/2}-1)$

e. $\frac{17^{3/2}}{12}$

$$\vec{v} = (1,2t) \qquad |\vec{v}| = \sqrt{1+4t^2} \qquad x = t$$

$$\int_{(0,0)}^{(\sqrt{2},2)} x\,ds = \int_0^{\sqrt{2}} x|\vec{v}|\,dt = \int_0^{\sqrt{2}} t\sqrt{1+4t^2}\,dt = \left[\frac{(1+4t^2)^{3/2}}{12}\right]_0^{\sqrt{2}} = \frac{13}{6}$$

10. Compute $\int_{(0,0)}^{(\sqrt{2},2)} \vec{\nabla}f\cdot d\vec{s}$ for $f = xy$ along the parabola $y = x^2$ parametrized as $\vec{r}(t) = (t,t^2)$.

a. 2

b. $2\sqrt{2}$ Correct Choice

c. 4

d. $4\sqrt{2}$

e. 8

$$\vec{v} = (1,2t) \qquad \vec{\nabla}f = (y,x) = (t^2,t) \qquad \vec{\nabla}f\cdot\vec{v} = t^2+2t^2 = 3t^2$$

$$\int_{(0,0)}^{(\sqrt{2},2)} \vec{\nabla}f\cdot d\vec{s} = \int_0^{\sqrt{2}} \vec{\nabla}f\cdot\vec{v}\,dt = \int_0^{\sqrt{2}} 3t^2\,dt = \left[t^3\right]_0^{\sqrt{2}} = 2\sqrt{2}$$

11. Compute $\int \vec{F} \cdot d\vec{s}$ for $\vec{F} = (-yz^2, xz^2, z^3)$ counterclockwise around the circle $x^2 + y^2 = 4$ with $z = 4$. HINT: Parametrize the circle.

a. 64

b. 128

c. 64π

d. 128π Correct Choice

e. 256π

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 4)$ $\vec{v} = (-2\sin\theta, 2\cos\theta, 0)$ $\vec{F}(\vec{r}(\theta)) = (-32\sin\theta, 32\cos\theta, 64)$

$\vec{F}(\vec{r}(\theta)) \cdot \vec{v} = 64\sin^2\theta + 64\cos^2\theta = 64$ $\int \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 64\, d\theta = 128\pi$

12. Compute $\iiint \vec{\nabla} \cdot \vec{F}\, dV$ for $\vec{F} = (xy^2, yz^2, zx^2)$ over the solid hemisphere $x^2 + y^2 + z^2 \le 25$ with $z \ge 0$.

a. $\frac{125}{3}\pi$

b. $\frac{250}{3}\pi$

c. $\frac{500}{3}\pi$

d. 1250π Correct Choice

e. 2500π

$\vec{\nabla} \cdot \vec{F} = y^2 + z^2 + x^2 = \rho^2$

$\iiint \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^5 \rho^2 \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi = \left[-\cos\varphi\right]_0^{\pi/2} (2\pi) \left[\frac{\rho^5}{5}\right]_0^5 = 2\pi 5^4 = 1250\pi$

Work Out: (Points indicated. Part credit possible. Show all work.)

13. (15 points) Find the volume of the largest box such the length plus twice the width plus three times the height is 36.

Maximize $V = LWH$ subject to the constraint $g = L + 2W + 3H = 36$.

Lagrange Multiplier Method

$\vec{\nabla}V = (WH, LH, LW) \qquad \vec{\nabla}g = (1,2,3)$

The Lagrange equations are $\vec{\nabla}V = \lambda\vec{\nabla}g$ or

$$\begin{matrix} WH = \lambda \\ LH = 2\lambda \\ LW = 3\lambda \end{matrix} \quad \Rightarrow \quad \begin{matrix} LH = 2WH \\ LW = 3WH \end{matrix} \quad \Rightarrow \quad \begin{matrix} W = \dfrac{L}{2} \\ H = \dfrac{L}{3} \end{matrix}$$

where we use $H \neq 0$ and $W \neq 0$ to give a maximum volume.

Plug into the constraint: $L + 2\left(\dfrac{L}{2}\right) + 3\left(\dfrac{L}{3}\right) = 3L = 36$

$L = 12 \qquad W = 6 \qquad H = 4 \qquad\qquad V = 12 \bullet 6 \bullet 4 = 288$

Eliminate a Variable Method:

$L = 36 - 2W - 3H$

$V = WH(36 - 2W - 3H) = 36WH - 2W^2H - 3WH^2$

$$\begin{matrix} V_W = 36H - 4WH - 3H^2 = 0 \\ V_H = 36W - 2W^2 - 6WH = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} 36 - 4W - 3H = 0 \\ 36 - 2W - 6H = 0 \end{matrix}$$

where we use $H \neq 0$ and $W \neq 0$ to give a maximum volume.

$$\begin{matrix} 4W + 3H = 36 \\ 2W + 6H = 36 \end{matrix} \quad \Rightarrow \quad \begin{matrix} 8W + 6H = 72 \\ 2W + 6H = 36 \end{matrix} \quad \Rightarrow \quad 6W = 36 \quad \Rightarrow \quad W = 6$$

$6H = 36 - 2W = 36 - 12 = 24 \quad \Rightarrow \quad H = 4$

$L = 36 - 2W - 3H = 36 - 12 - 12 = 12 \qquad\qquad V = 12 \bullet 6 \bullet 4 = 288$

14. (10 points) Find the area of the surface $\vec{R}(p,q) = (p, p + q^2, p - q^2)$ for $0 \le p \le 3$ and $0 \le q \le 2$.

$$\begin{matrix} \\ \vec{e}_p = \\ \vec{e}_q = \end{matrix} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\ 1 & 1 & 1\ ) \\ (\ 0 & 2q & -2q\ ) \end{vmatrix}$$

$\vec{N} = \vec{e}_p \times \vec{e}_q = \hat{\imath}(-2q - 2q) - \hat{\jmath}(-2q) + \hat{k}(2q) = (-4q, 2q, 2q)$

$\left|\vec{N}\right| = \sqrt{16q^2 + 4q^2 + 4q^2} = \sqrt{24}\,q$

$A = \iint 1\,dS = \iint \left|\vec{N}\right|\,dp\,dq = \int_0^2\int_0^3 \sqrt{24}\,q\,dp\,dq = \sqrt{24}\left[p\right]_0^3\left[\dfrac{q^2}{2}\right]_0^2 = 6\sqrt{24} = 12\sqrt{6}$

15. (15 points) Compute $\iint_P \vec{\nabla}\times\vec{F}\cdot d\vec{S}$ for $\vec{F} = (-yz^2, xz^2, z^3)$ over the paraboloid $z = x^2 + y^2$ with $z \le 4$, oriented **up and in**, and parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz^2 & xz^2 & z^3 \end{vmatrix} = \hat{\imath}(0-2xz) - \hat{\jmath}(0--2yz) + \hat{k}(z^2--z^2) = (-2xz, -2yz, 2z^2)$$

$$\left(\vec{\nabla}\times\vec{F}\right)\left(\vec{R}(r,\theta)\right) = (-2r^3\cos\theta, -2r^3\sin\theta, 2r^4)$$

$$\begin{matrix} & \\ \vec{e}_r = \\ \vec{e}_\theta = \end{matrix} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\ \cos\theta & \sin\theta & 2r) \\ (-r\sin\theta & r\cos\theta & 0\ ) \end{vmatrix}$$

$$\vec{N} = \vec{e}_r\times\vec{e}_\theta = \hat{\imath}(0-2r^2\cos\theta) - \hat{\jmath}(0--2r^2\sin\theta) + \hat{k}(r\cos^2\theta--r\sin^2\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$\vec{N}$ is correctly oriented up and in.

$$\vec{\nabla}\times\vec{F}\cdot\vec{N} = 4r^5\cos^2\theta + 4r^5\sin^2\theta + 2r^5 = 6r^5$$

$$z = r^2 \le 4 \quad \Rightarrow \quad r \le 2$$

$$\iint_P \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^2 \vec{\nabla}\times\vec{F}\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^2 6r^5\,dr\,d\theta = 2\pi[r^6]_0^2 = 128\pi$$

16. (5 points) Mark the Project that you worked on and then answer the questions in 1 or 2 sentences.

___Cauchy's Gradient Method
How did the procedure for finding a maximum differ from the procedure for finding a minimum?

___Seeing a Blimp
Approximately how many degrees wide was the blimp? (1 digit accuracy)

___The Trash Dumpster
Draw the basic shape of your dumpster. Label your variables. Label your hinges and welds.

___Locating an Apartment
What was the geometrical condition satisfied by the location of the apartment?
A diagram may help.

___Rectangles and Triangles
Where were the lines in the rectangle which minimized the sum of the areas?
Where were the lines in the triangle which minimized the sum of the areas?
Two diagrams may help.

___Exact Gradient Method
What is the relation between the velocity of your curve and the gradient of the function to find a maximum? What is the relation to find a minimum?