Name________________________

1-11	/44	14	/10
12	/15	15	/20
13	/15	Total	/104

MATH 253 Final Fall 2009

Section 501,503 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. Find the point where the line $x = 1 + 2t, \quad y = 8 - 3t, \quad z = 2 - 2t$ intersects the plane $x - y + z = 1$. At this point $x + y + z =$

a. 9

b. 5 Correct Choice

c. 2

d. 1

e. 0

$x - y + z = (1 + 2t) - (8 - 3t) + (2 - 2t) = 3t - 5 = 1 \qquad t = 2$

$x = 5, \quad y = 2, \quad z = -2, \qquad x + y + z = 5$

2. Find the plane tangent to the graph of $z = \cos(x + 2y)$ at the point $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. The z-intercept is

a. 0

b. $\frac{\pi}{6}$

c. $\frac{\pi}{3}$

d. $\frac{\pi}{2}$ Correct Choice

e. π

$f(x,y) = \cos(x + 2y) \qquad f_x(x,y) = -\sin(x + 2y) \qquad f_y(x,y) = -2\sin(x + 2y)$

$f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{2}\right) = 0 \qquad f_x\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{2}\right) = -1 \qquad f_y\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = -2\sin\left(\frac{\pi}{2}\right) = -2$

Tan plane: $z = -1\left(x - \frac{\pi}{6}\right) - 2\left(y - \frac{\pi}{6}\right) = -x - 2y + \frac{\pi}{6} + \frac{\pi}{3} = -x - 2y + \frac{\pi}{2}$

z-intercept $= \frac{\pi}{2}$

3. Find the plane tangent to the surface $\frac{x}{z} + \frac{z}{y} = 5$ at the point $P = (6,1,3)$. The z-intercept is

a. $(0,0,0)$ Correct Choice

b. $(0,0,-5)$

c. $(0,0,5)$

d. $(0,0,-10)$

e. $(0,0,10)$

$F = \frac{x}{z} + \frac{z}{y} \qquad \vec{\nabla}F = \left(\frac{1}{z}, -\frac{z}{y^2}, -\frac{x}{z^2} + \frac{1}{y}\right) \qquad \vec{N} = \vec{\nabla}F\Big|_{(6,1,3)} = \left(\frac{1}{3}, -3, -\frac{6}{9} + 1\right) = \left(\frac{1}{3}, -3, \frac{1}{3}\right)$

$\vec{N} \cdot X = \vec{N} \cdot P \qquad \frac{1}{3}x - 3y + \frac{1}{3}z = \frac{1}{3}(6) - 3(1) + \frac{1}{3}(3) = 0$

Intersects the z-axis when $x = y = 0$. So $z = 0$.

4. A circuit has two resistors $R_1 = 200\ \Omega$ and $R_2 = 300\ \Omega$ in parallel. The net resistance R satisfies $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$. If R_1 is increasing at $2\ \Omega/\text{sec}$ and R_2 is decreasing at $9\ \Omega/\text{sec}$ at what rate is R changing?

a. $\dfrac{9}{50}\ \Omega/\text{sec}$

b. $\dfrac{18}{25}\ \Omega/\text{sec}$

c. $-\dfrac{9}{50}\ \Omega/\text{sec}$

d. $-\dfrac{9}{25}\ \Omega/\text{sec}$

e. $-\dfrac{18}{25}\ \Omega/\text{sec}$ Correct Choice

$$\frac{1}{R} = \frac{1}{200} + \frac{1}{300} = \frac{300+200}{200 \cdot 300} = \frac{1}{120} \qquad R = 120$$

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1^2}\frac{dR_1}{dt} - \frac{1}{R_2^2}\frac{dR_2}{dt} \qquad \frac{dR}{dt} = \frac{R^2}{R_1^2}\frac{dR_1}{dt} + \frac{R^2}{R_2^2}\frac{dR_2}{dt} = \frac{120^2}{200^2}2 - \frac{120^2}{300^2}9 = -\frac{18}{25}$$

5. Ham Duet is flying the Millenium Eagle through a galactic dust storm. Currently, his position is $P = (10,-20,30)$ and his velocity is $\vec{v} = (4,-12,3)$. He measures that currently the dust density is $\rho = 500$ and its gradient is $\vec{\nabla}\rho = (-2,1,2)$. Find the current rate of change of the dust density as seen by Ham.

a. 514

b. 486

c. 28

d. 14

e. -14 Correct Choice

$$\vec{\nabla}_{\vec{v}}\rho = \vec{v} \cdot \vec{\nabla}\rho = (4,-12,3) \cdot (-2,1,2) = -8 - 12 + 6 = -14$$

6. Under the same conditions as in #5, in what **unit** vector direction should Ham travel to **decrease** the dust density as quickly as possible?

a. $(-2,1,2)$

b. $(2,-1,-2)$

c. $\left(\dfrac{2}{3},\dfrac{-1}{3},\dfrac{-2}{3}\right)$ Correct Choice

d. $\left(\dfrac{4}{13},\dfrac{-12}{13},\dfrac{3}{13}\right)$

e. $\left(\dfrac{-4}{13},\dfrac{12}{13},\dfrac{-3}{13}\right)$

$$\hat{u} = \frac{-\vec{\nabla}\rho}{\left|\vec{\nabla}\rho\right|} = \frac{-(-2,1,2)}{\sqrt{4+1+4}} = \left(\frac{2}{3},\frac{-1}{3},\frac{-2}{3}\right)$$

7. The point $(1,-2)$ is a critical point of the function $f = x^2y^2 + \frac{8}{x} - \frac{16}{y}$. Use the Second Derivative Test to classify the point.

a. Local Minimum Correct Choice

b. Local Maximum

c. Inflection Point

d. Saddle Point

e. Test Fails

$f = x^2y^2 + \frac{8}{x} - \frac{16}{y} \qquad f_x = 2xy^2 - \frac{8}{x^2} \qquad f_y = 2x^2y + \frac{16}{y^2}$

$f_{xx} = 2y^2 + \frac{16}{x^3} = 24 > 0 \qquad f_{yy} = 2x^2 - \frac{32}{y^3} = 6 > 0 \qquad f_{xy} = 4xy = -8$

$D = 144 - 64 = 80 > 0$ Minimum

8. Compute $\oint \vec{F} \cdot d\vec{s}$ counterclockwise around the circle $x^2 + y^2 = 4$ for $\vec{F} = (x^4 - y^3, y^4 + x^3)$.

HINT: Use the Fundamental Theorem of Calculus for Curves or Green's Theorem.

a. 0

b. 8π

c. 16π

d. 24π Correct Choice

e. 32π

By Green's Theorem: $\vec{F} = (P,Q) \qquad P = x^4 - y^3 \qquad Q = y^4 + x^3$

$$\oint \vec{F} \cdot d\vec{s} = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint (3x^2 - -3y^2)\,dx\,dy = \int_0^{2\pi} \int_0^2 3r^2\, r\,dr\,d\theta = 2\pi \left[\frac{3r^4}{4} \right]_0^2 = 24\pi$$

9. The surface of an apple A may be given in spherical coordinates by $\rho = 1 - \cos\varphi$ and may be parametrized by $R(\phi,\theta) = ((1-\cos\varphi)\sin\varphi\cos\theta, (1-\cos\varphi)\sin\varphi\sin\theta, (1-\cos\varphi)\cos\varphi)$.

Compute $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the apple with outward normal for $\vec{F} = (xyz^2, yzx^2, zxy^2)$.

HINT: Use Stokes' Theorem or Gauss' Theorem.

a. 0 Correct Choice

b. 4π

c. 12π

d. $\frac{32}{3}\pi$

e. $\frac{64}{3}\pi$

By Stokes' Theorem, $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint \vec{F} \cdot d\vec{s} = 0$ because there is no boundary curve.

By Gauss' Theorem, $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iiint \vec{\nabla} \cdot \vec{\nabla} \times \vec{F}\,dV = 0$ because $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$.

10. Find the mass of the spiral $\vec{r}(\theta) = (\theta\cos\theta, \theta\sin\theta)$ for $0 \le \theta \le 6\pi$ if the linear density is $\rho = \sqrt{x^2 + y^2}$.

a. $\frac{1}{2}\ln\left(6\pi + \sqrt{1 + 36\pi^2}\right) + 3\pi\sqrt{1 + 36\pi^2}$

b. $\frac{1}{2}\ln\left(6\pi + \sqrt{1 + 6\pi}\right) - 3\pi\sqrt{1 + 6\pi}$

c. $\frac{1}{2}\ln\left(6\pi + \sqrt{1 + 6\pi}\right) + 3\pi\sqrt{1 + 6\pi}$

d. $\frac{1}{3}(1 + 36\pi^2)^{3/2} - \frac{1}{3}$ Correct Choice

e. $\frac{1}{3}(1 + 6\pi)^{3/2} - \frac{1}{3}$

$$\vec{v} = (\cos\theta - \theta\sin\theta, \sin\theta + \theta\cos\theta)$$

$$|\vec{v}| = \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2} = \sqrt{1 + \theta^2}$$

$$\rho = \sqrt{\theta^2\cos^2\theta + \theta^2\sin^2\theta} = \theta$$

$$M = \int \rho\, ds = \int_0^{6\pi} \theta\sqrt{1 + \theta^2}\, d\theta = \left[\frac{1}{3}(1 + \theta^2)^{3/2}\right]_0^{6\pi} = \frac{1}{3}(1 + 36\pi^2)^{3/2} - \frac{1}{3}$$

11. Use Stokes' Theorem to compute $\oint \vec{F} \cdot d\vec{s}$ around the triangle with vertices $A = (2,0,0)$, $B = (0,3,0)$ and $C = (0,0,6)$, traversed from A to B to C to A for $\vec{F} = (y,z,x)$.

Note: The plane of the triangle may be parametrized as $\vec{R}(x,y) = (x, y, 6 - 3x - 2y)$.

a. -24

b. -18 Correct Choice

c. 12

d. 18

e. 24

$$\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_x = & (1 & 0 & -3) \\ \vec{e}_y = & (0 & 1 & -2) \end{vmatrix} \qquad \vec{N} = \hat{\imath}(3) - \hat{\jmath}(-2) + \hat{k}(1) = (3,2,1)$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & z & x \end{vmatrix} = \hat{\imath}(-1) - \hat{\jmath}(1) + \hat{k}(-1) = (-1,-1,-1)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = -3 - 2 - 1 = -6$$

$$\oint_{\partial T} \vec{F} \cdot d\vec{s} = \iint_T \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_0^2 \int_0^{3 - \frac{3}{2}x} -6\, dy\, dx = -6\,\text{Area} = -6\left(\frac{1}{2} 2 \cdot 3\right) = -18$$

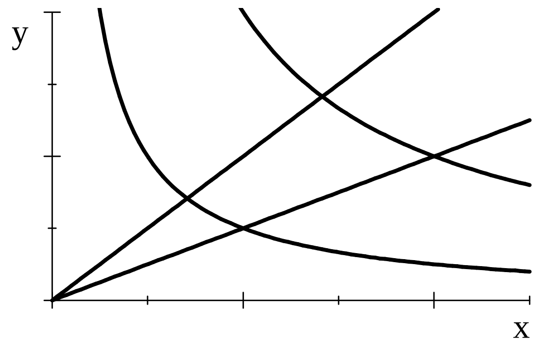
Work Out: (Points indicated. Part credit possible. Show all work.)

12. (15 points) Compute $\iint_D y^2\,dx\,dy$ over the "diamond shaped" region D in the first quadrant bounded by the hyperbolas

$y = \frac{1}{x}$ and $y = \frac{4}{x}$

and the lines

$y = x$ and $y = 2x$



HINT: Use the coordinates $u = xy$, $v = \frac{y}{x}$. Solve for x and y.

$y = xv \qquad u = x^2 v \qquad x^2 = \frac{u}{v} \qquad \boxed{x = \sqrt{\frac{u}{v}} = u^{1/2} v^{-1/2}} \qquad y = \sqrt{\frac{u}{v}}\,v \qquad \boxed{y = \sqrt{uv} = u^{1/2} v^{1/2}}$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix} \frac{1}{2}u^{-1/2}v^{-1/2} & \frac{1}{2}u^{-1/2}v^{1/2} \\ -\frac{1}{2}u^{1/2}v^{-3/2} & \frac{1}{2}u^{1/2}v^{-1/2} \end{matrix}\right\| = \left|\frac{1}{4v} - -\frac{1}{4v}\right| = \frac{1}{2v}$$

Integrand: $y^2 = uv$ Boundaries are:

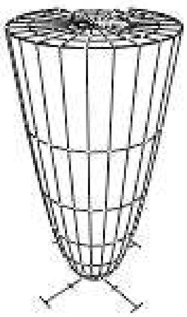
$y = \frac{1}{x} \Rightarrow xy = 1 \Rightarrow u = 1 \qquad y = \frac{4}{x} \Rightarrow xy = 4 \Rightarrow u = 4$

$y = x \Rightarrow \frac{y}{x} = 1 \Rightarrow v = 1 \qquad y = 2x \Rightarrow \frac{y}{x} = 2 \Rightarrow v = 2$

$$\iint_D y^2\,dx\,dy = \int_1^2\int_1^4 uv\frac{1}{2v}\,du\,dv = \frac{1}{2}\int_1^2\int_1^4 u\,du\,dv = \frac{1}{2}[v]_1^2\left[\frac{u^2}{2}\right]_1^4 = \frac{15}{4}$$

13. (15 points) Find the volume and z-component of the centroid (center of mass with $\rho = 1$) of the solid between the surfaces

$$z = (x^2 + y^2)^{3/2} \quad \text{and} \quad z = 8.$$



In cylindrical coordinates: $r^3 \le z \le 8$. So $0 \le r \le 2$.

$$V = \int_0^{2\pi}\int_0^2\int_{r^3}^8 r\,dz\,dr\,d\theta = 2\pi\int_0^2\Big[rz\Big]_{z=r^3}^8 dr = 2\pi\int_0^2 (8r - r^4)\,dr = 2\pi\Big[4r^2 - \frac{r^5}{5}\Big]_0^2$$

$$= 2\pi\Big(16 - \frac{32}{5}\Big) = 32\pi\Big(1 - \frac{2}{5}\Big) = \frac{96\pi}{5}$$

$$M_{xy} = \int_0^{2\pi}\int_0^2\int_{r^3}^8 zr\,dz\,dr\,d\theta = 2\pi\int_0^2\Big[r\frac{z^2}{2}\Big]_{z=r^3}^8 dr = \pi\int_0^2 (64r - r^7)\,dr = \pi\Big[32r^2 - \frac{r^8}{8}\Big]_0^2$$

$$= \pi(128 - 32) = 96\pi$$

$$\bar{z} = \frac{M_{xy}}{V} = 96\pi\frac{5}{96\pi} = 5$$

14. (10 points) Find the point in the first octant on the graph of $xy^2z^4 = 32$ which is closest to the origin.

HINTS: What is the square of the distance from a point to the origin? Lagrange multipliers are easier.

Minimize $f = x^2 + y^2 + z^2$ subject to $g = xy^2z^4 = 32$.

Method 1: Lagrange multipliers:

$$\vec{\nabla}f = (2x, 2y, 2z) \qquad \vec{\nabla}g = (y^2z^4, 2xyz^4, 4xy^2z^3)$$

$$\vec{\nabla}f = \lambda\vec{\nabla}g \quad \Rightarrow \quad 2x = \lambda y^2z^4, \quad 2y = \lambda 2xyz^4, \quad 2z = \lambda 4xy^2z^3$$

$$\lambda = \frac{2x}{y^2z^4} = \frac{1}{xz^4} = \frac{1}{2xy^2z^2} \quad \Rightarrow \quad 2x^2 = y^2, \quad 4x^2 = z^2 \quad \Rightarrow \quad y = \sqrt{2}\,x, \quad z = 2x$$

$$32 = xy^2z^4 = x\big(\sqrt{2}\,x\big)^2(2x)^4 = 32x^7 \quad \Rightarrow \quad x = 1 \quad y = \sqrt{2} \quad z = 2$$

Method 2: Eliminate a variable:

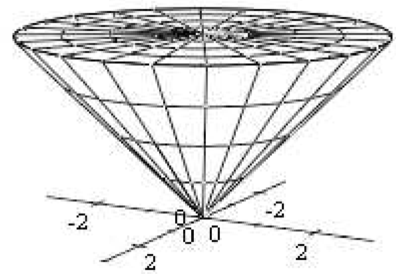
$$x = \frac{32}{y^2z^4} \qquad f = \frac{2^{10}}{y^4z^8} + y^2 + z^2$$

$$f_y = -\frac{2^{12}}{y^5z^8} + 2y = 0 \qquad f_z = -\frac{2^{13}}{y^4z^9} + 2z = 0 \quad \Rightarrow \quad y^6z^8 = 2^{11} \qquad y^4z^{10} = 2^{12}$$

$$\Rightarrow \quad 2 = \frac{y^4z^{10}}{y^6z^8} = \frac{z^2}{y^2} \quad \Rightarrow \quad z = \sqrt{2}\,y \quad \Rightarrow \quad y^6\big(\sqrt{2}\,y\big)^8 = 2^{11} \quad \Rightarrow \quad y^{14} = 2^7$$

$$\Rightarrow \quad y = \sqrt{2} \qquad z = 2 \qquad x = \frac{32}{y^2z^4} = \frac{2^5}{2 \cdot 2^4} = 1$$

15. (20 points) Verify Gauss' Theorem $\iiint\limits_V \vec\nabla \cdot \vec F\, dV = \iint\limits_{\partial V} \vec F \cdot d\vec S$



for the vector field $\vec F = (xy^2, yx^2, z^3)$ and the volume above the cone $z = \sqrt{x^2+y^2}$ and below the plane $z = 2$.

Use the following steps:

a. Compute the volume integral:

$$\vec\nabla \cdot \vec F = y^2 + x^2 + 3z^2 = r^2 + 3z^2$$

$$\iiint\limits_V \vec\nabla \cdot \vec F\, dV = \int_0^{2\pi}\int_0^2\int_r^2 (r^2+3z^2)r\,dz\,dr\,d\theta = 2\pi\int_0^2 \left[r^3 z + z^3 r\right]_{z=r}^2 dr = 2\pi\int_0^2 (2r^3+8r)-(2r^4)\,dr$$

$$= 2\pi\left[\frac{r^4}{2} + 4r^2 - \frac{2r^5}{5}\right]_0^2 = 2^5\pi\left(\frac{1}{2}+1-\frac{4}{5}\right) = 2^5\pi\left(\frac{5+10-8}{10}\right) = \frac{7\cdot 2^4\pi}{5} = \frac{112}{5}\pi$$

b. Compute the surface integral over the disk using the parametrization

$$\vec R(r,\theta) = (\ r\cos\theta\ ,\ r\sin\theta\ ,\ 2\ ):$$

$$\vec e_r = (\ \cos\theta\ ,\ \sin\theta\ ,\ 0\ )$$

$$\vec e_\theta = (\ -r\sin\theta\ ,\ r\cos\theta\ ,\ 0\ )$$

$$\vec N = \hat\imath(0) - \hat\jmath(0) + \hat k(r\cos^2\theta + r\sin^2\theta) = (0,0,r)$$

$$\vec F\left(\vec R(r,\theta)\right) = (xy^2, yx^2, z^3) = (r^3\cos\theta\sin^2\theta, r^3\sin\theta\cos^2\theta, 8)$$

$$\iint\limits_D \vec F\cdot d\vec S = \iint\limits_C \vec F\cdot\vec N\,dr\,d\theta = \int_0^{2\pi}\int_0^2 8r\,dr\,d\theta = 2\pi\left[4r^2\right]_0^2 = 32\pi$$

c. Compute the surface integral over the cone using the parametrization

$$\vec R(r,\theta) = (\ r\cos\theta\ ,\ r\sin\theta\ ,\ r\ ):$$

$$\vec e_r = (\ \cos\theta\ ,\ \sin\theta\ ,\ 1\ )$$

$$\vec e_\theta = (\ -r\sin\theta\ ,\ r\cos\theta\ ,\ 0\ )$$

$$\vec N = \hat\imath(-r\cos\theta) - \hat\jmath(r\sin\theta) + \hat k(r\cos^2\theta + r\sin^2\theta) = (-r\cos\theta, -r\sin\theta, r)$$

Reverse orientation: $\vec N = (r\cos\theta, r\sin\theta, -r)$

$$\vec F\left(\vec R(r,\theta)\right) = (xy^2, yx^2, z^3) = (r^3\cos\theta\sin^2\theta, r^3\sin\theta\cos^2\theta, r^3)$$

$$\iint\limits_C \vec F\cdot d\vec S = \iint\limits_C \vec F\cdot\vec N\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (r^4\cos^2\theta\sin^2\theta + r^4\sin^2\theta\cos^2\theta - r^4)\,dr\,d\theta$$

$$= \int_0^2 r^4\,dr\int_0^{2\pi}(2\sin^2\theta\cos^2\theta - 1)\,d\theta = \left[\frac{r^5}{5}\right]_0^2\int_0^{2\pi}\left(\frac{\sin^2 2\theta}{2} - 1\right)d\theta =$$

$$= \frac{32}{5}\int_0^{2\pi}\left(\frac{1-\cos 4\theta}{4} - 1\right)d\theta = \frac{32}{5}\left(\frac{2\pi}{4} - 2\pi\right) = -\frac{48}{5}\pi$$

d. Compute the surface integral over the total boundary:

$$\iint\limits_{\partial V} \vec F\cdot d\vec S = \iint\limits_D \vec F\cdot d\vec S + \iint\limits_C \vec F\cdot d\vec S = 32\pi - \frac{48}{5}\pi = \frac{112}{5}\pi$$