MATH 253 EXAM 1 Spring 1998
Sections 501-503 P. Yasskin

Problems 1 – 3: Find the plane tangent to the the graph of the function $f(x,y) = \dfrac{36}{1+x^2+y^2}$ at the point $(x,y) = (1,2)$. Write the equation of the plane in the form $z = Ax + By + C$ and find the values of A, B and C in problems 1, 2 and 3:

$f(x,y) = \dfrac{36}{1+x^2+y^2}$ $\quad f(1,2) = \dfrac{36}{1+1^2+2^2} = 6$

$f_x(x,y) = \dfrac{-36 \bullet 2x}{1+x^2+y^2}$ $\quad f_x(1,2) = \dfrac{-36 \bullet 2 \bullet 1}{1+1^2+2^2} = -2$ $\quad A = -2$

$f_y(x,y) = \dfrac{-36 \bullet 2y}{1+x^2+y^2}$ $\quad f_y(1,2) = \dfrac{-36 \bullet 2 \bullet 2}{1+1^2+2^2} = -4$ $\quad B = -4$

$z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 6 - 2(x-1) - 4(y-2) = -2x - 4y + 16$ $\quad C = 16$

1. (3 points) $A =$

- **a**. -4
- **b**. -2 correctchoice
- **c**. 0
- **d**. 2
- **e**. 4

2. (3 points) $B =$

- **a**. -4 correctchoice
- **b**. -2
- **c**. 0
- **d**. 2
- **e**. 4

3. (3 points) $C =$

- **a**. -4
- **b**. 3
- **c**. 6
- **d**. 9
- **e**. 16 correctchoice

4. (3 points) If the function $f(x,y) = \dfrac{36}{1+x^2+y^2}$ represents the height of a mountain and you are at the point $(x,y) = (1,2)$, in what direction should you walk to go directly **down** hill?

- **a**. $(-4,-2)$
- **b**. $(-2,-4)$ $\quad \vec{\nabla} f = (-2,-4)$ points up hill.
- **c**. $(4,2)$
- **d**. $(2,4)$ correctchoice $\quad -\vec{\nabla} f = (2,4)$ points **down** hill.
- **e**. None of these

Problems 5 – 7: Find the plane tangent to the the graph of the equation $xe^z + ze^{xy} = 2$ at the point $(x,y,z) = (0,1,2)$. Write the equation of the plane in the form $z = Ax + By + C$ and find the values of A, B and C in problems 5, 6 and 7:

$f(x,y,z) = xe^z + ze^{xy}$ $\quad \vec{\nabla} f = (e^z + zye^{xy}, zxe^{xy}, xe^z + e^{xy})$ $\quad X = (x,y,z)$ $\quad P = (0,1,2)$

$\vec{N} = \vec{\nabla} f\Big|_{(0,1,2)} = (e^2 + 2e^0, 0, e^0) = (e^2 + 2e^0, 0, 1)$

$N \bullet X = N \bullet P$ $\quad (e^2 + 2)x + z = 2$ $\quad z = -(e^2 + 2)x + 2$

5. (3 points) $A =$ $\quad A = -(e^2 + 2)$

- **a**. $-2 - e$
- **b**. $2 + e$
- **c**. $-2 - e^2$ correctchoice
- **d**. $2 - e^2$
- **e**. 0

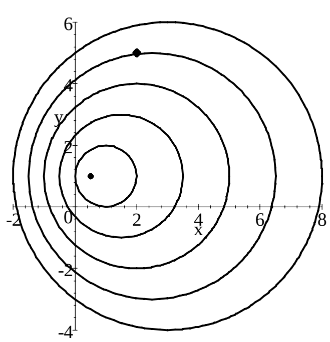
6. (3 points) $B =$ $\quad B = 0$

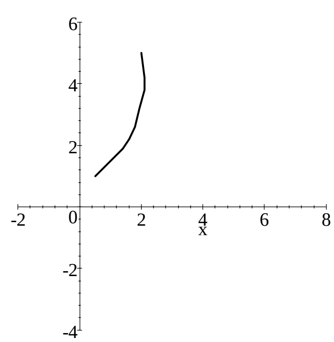
- **a**. $-2 - e$
- **b**. $2 + e$
- **c**. $-2 - e^2$
- **d**. $2 - e^2$
- **e**. 0 correctchoice

7. (3 points) $C =$ $\quad C = 2$

- **a**. 2 correctchoice
- **b**. $-e$
- **c**. $\frac{1}{e}$
- **d**. $\frac{2}{e}$
- **e**. e^2

8. (5 points) Below is the contour plot of a function $f(x,y)$. If you start at the point $(2,5)$ and move along a curve whose tangent vector is always $\vec{v} = \vec{\nabla} f$, draw the curve in the plot.





The curve starts at $(2,5)$ ends at $(.5,1)$ and is perpendicular to each level curve (circle). The curve should be drawn in the same plot.

9. (12 points) Find all critical points of the function $f(x,y) = 1 + 2xy - x^2 - \frac{1}{9}y^3$ and classify each as a local maximum, a local minimum or a saddle point.

$f_x = 2y - 2x = 0 \qquad f_y = 2x - \frac{1}{3}y^2 = 0 \qquad \Rightarrow \qquad x = y \qquad 2y - \frac{1}{3}y^2 = 0 \qquad y\left(2 - \frac{1}{3}y\right) = 0$

So $y = 0$ or $y = 6$ and $x = y$. So the critical points are $(0,0)$ and $(6,6)$.

$f_{xx} = -2 \qquad f_{yy} = -\frac{2}{3}y \qquad f_{xy} = 2 \qquad D = f_{xx}f_{yy} - f_{xy}{}^2 = (-2)\left(-\frac{2}{3}y\right) - (2)^2 = \frac{4}{3}y - 4$

x	y	f_{xx}	D		
0	0	-2	-4	saddle	since $D < 0$
6	6	-2	4	local maximum	since $D > 0$ and $f_{xx} < 0$

10. (12 points) Find the point on the paraboloid $z - \frac{1}{2}x^2 - \frac{1}{2}y^2 = 0$ which is closest to the point $(1,2,1)$.

Minimize the square of the distance from the general point (x,y,z) to the point $(1,2,1)$:

$$f = (x-1)^2 + (y-2)^2 + (z-1)^2$$

subject to the constraint that the general point (x,y,z) is on the paraboloid

$$g = z - \frac{1}{2}x^2 - \frac{1}{2}y^2 = 0$$

Method 1: **Lagrange Multipliers**.

$\nabla f = (2(x-1), 2(y-2), 2(z-1)) \qquad \nabla g = (-x, -y, 1) \qquad \nabla f = \lambda \nabla g$

$2(x-1) = -\lambda x \qquad \lambda = \frac{2(x-1)}{-x} = -2 + \frac{2}{x} \qquad$ #1

$2(y-2) = -\lambda y \qquad \lambda = \frac{2(y-2)}{-y} = -2 + \frac{4}{y} \qquad$ #2

$2(z-1) = \lambda \qquad \lambda = 2(z-1) = 2z - 2 \qquad$ #3

Equate #1 and #2: $\frac{2}{x} = \frac{4}{y} \qquad y = 2x$

Equate #3 and #1: $2z = \frac{2}{x} \qquad z = \frac{1}{x}$

Plug both into the constraint: $\frac{1}{x} - \frac{1}{2}x^2 - \frac{1}{2}(2x)^2 = 0 \qquad \frac{1}{x} = \frac{5}{2}x^2 \qquad x^3 = \frac{2}{5}$

$x = \sqrt[3]{\frac{2}{5}} \qquad y = 2\sqrt[3]{\frac{2}{5}} \qquad z = \sqrt[3]{\frac{5}{2}}$

Method 2: **Eliminate a Variable**.

$z = \frac{1}{2}x^2 + \frac{1}{2}y^2 \qquad f = (x-1)^2 + (y-2)^2 + \left(\frac{1}{2}x^2 + \frac{1}{2}y^2 - 1\right)^2$

$f_x = 2(x-1) + 2\left(\frac{1}{2}x^2 + \frac{1}{2}y^2 - 1\right)(x) = 0 \qquad \left(\frac{1}{2}x^2 + \frac{1}{2}y^2 - 1\right) = -\frac{2(x-1)}{2x} = -1 + \frac{1}{x} \qquad$ #1

$f_y = 2(y-2) + 2\left(\frac{1}{2}x^2 + \frac{1}{2}y^2 - 1\right)(x) = 0 \qquad \left(\frac{1}{2}x^2 + \frac{1}{2}y^2 - 1\right) = -\frac{2(y-2)}{2y} = -1 + \frac{2}{y} \qquad$ #2

Equate #1 and #2: $\frac{1}{x} = \frac{2}{y} \qquad y = 2x$

Plug back into #1: $\frac{1}{2}x^2 + \frac{1}{2}(2x)^2 - 1 = -1 + \frac{1}{x} \qquad \frac{5}{2}x^2 = \frac{1}{x} \qquad x^3 = \frac{2}{5}$

$x = \sqrt[3]{\frac{2}{5}} \qquad y = 2\sqrt[3]{\frac{2}{5}} \qquad z = \frac{1}{2}x^2 + \frac{1}{2}y^2 = \frac{1}{2}x^2 + \frac{1}{2}(2x)^2 = \frac{5}{2}x^2 = \frac{5}{2}\sqrt[3]{\frac{2}{5}}^{\,2} = \sqrt[3]{\frac{5}{2}}$