MATH 253 | EXAM 2 | Spring 1998
Sections 501-503 | SOLUTIONS | P. Yasskin

1. (5 points) Compute $\int_0^1 \int_{x^2}^x x^2 y \, dy \, dx$.

a. $\dfrac{1}{70}$

b. $\dfrac{1}{35}$ correctchoice

c. $\dfrac{2}{35}$

d. $\dfrac{1}{14}$

e. $\dfrac{1}{7}$

$$\int_0^1 \int_{x^2}^x x^2 y \, dy \, dx = \int_0^1 \left[x^2 \frac{y^2}{2} \right]_{y=x^2}^x dx = \int_0^1 \left[x^2 \frac{x^2}{2} \right] - \left[x^2 \frac{(x^2)^2}{2} \right] dx$$
$$= \int_0^1 \frac{x^4}{2} - \frac{x^6}{2} \, dx = \left[\frac{x^5}{10} - \frac{x^7}{14} \right]_0^1 = \frac{1}{10} - \frac{1}{14} = \frac{7-5}{70} = \frac{1}{35}$$

2. (5 points) Find the volume below the plane $z = 4x + 10y$ above the region between the parabola $y = x^2$ and the line $y = x$.

a. 1 correctchoice

b. 2

c. 3

d. 4

e. 5

$$\int_0^1 \int_{x^2}^x (4x + 10y) \, dy \, dx = \int_0^1 [4xy + 5y^2]_{y=x^2}^x \, dx = \int_0^1 [4x^2 + 5x^2] - [4x^3 + 5x^4] \, dx$$
$$= \int_0^1 (9x^2 - 4x^3 - 5x^4) \, dx = [3x^3 - x^4 - x^5]_0^1 = 3 - 1 - 1 = 1$$

3. (5 points) Compute $\int_0^1 \int_0^x \int_0^{x+y} x \, dz \, dy \, dx$.

a. $\dfrac{1}{8}$

b. $\dfrac{1}{4}$

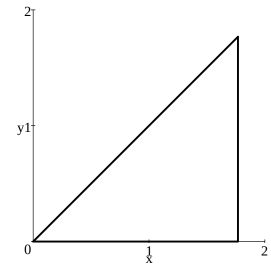
c. $\dfrac{3}{8}$ correctchoice

d. $\dfrac{1}{2}$

e. $\dfrac{5}{8}$

$$\int_0^1 \int_0^x \int_0^{x+y} x \, dz \, dy \, dx = \int_0^1 \int_0^x [xz]_{z=0}^{x+y} \, dy \, dx = \int_0^1 \int_0^x (x^2 + xy) \, dy \, dx = \int_0^1 \left[x^2 y + x \frac{y^2}{2} \right]_0^x dx$$
$$= \int_0^1 \left(x^3 + \frac{x^3}{2} \right) dx = \int_0^1 \frac{3}{2} x^3 \, dx = \left[\frac{3}{2} \bullet \frac{x^4}{4} \right]_0^1 = \frac{3}{8}$$

4. (5 points) Compute $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\ dx\ dy$.



a. 1 correctchoice

b. 2

c. 3

d. 4

e. Cannot be computed.

$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \sin(x^2)\ dx\ dy = \int_0^{\sqrt{\pi}} \int_0^{x} \sin(x^2)\ dy\ dx = \int_0^{\sqrt{\pi}} \sin(x^2)\ x\ dx = \left[\frac{\cos(x^2)}{-2} \right]_0^{\sqrt{\pi}}$$

$$= \left[\frac{\cos(\pi)}{-2} \right] - \left[\frac{\cos(0)}{-2} \right] = \frac{1}{2} + \frac{1}{2} = 1$$

5. (5 points) Reversing the order of integration gives $\int_0^1 \int_{x^2}^{x} f(x,y)\ dy\ dx =$

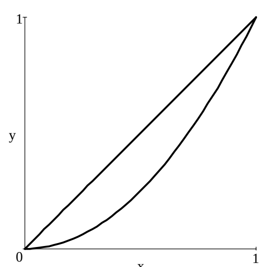
a. $\int_0^1 \int_{x^2}^{x} f(x,y)\ dx\ dy$

b. $\int_{x^2}^{x} \int_0^1 f(x,y)\ dx\ dy$

c. $\int_0^1 \int_y^{\sqrt{y}} f(x,y)\ dx\ dy$ correctchoice

d. $\int_0^1 \int_{y^2}^{y} f(x,y)\ dx\ dy$

e. $\int_0^1 \int_{\sqrt{y}}^{y} f(x,y)\ dx\ dy$



The first integral has ranges

$0 \le x \le 1 \qquad x^2 \le y \le x$

The new integral has ranges

$0 \le y \le 1 \qquad y \le x \le \sqrt{y}$

6. (5 points) Compute $\iint\limits_D e^{-x^2-y^2}\ dx\ dy$. over the disk $D = \{(x,y)\ |\ x^2+y^2 \le 4\}$.

a. $\frac{\pi}{2}(e^4-1)$

b. $\frac{\pi}{2}(1-e^{-4})$

c. $\pi(e^4-1)$

d. $\pi(1-e^{-4})$ correctchoice

e. Cannot be computed.

Convert to polar coordinates:

$$\iint e^{-x^2-y^2}\ dx\ dy = \int_0^{2\pi}\int_0^2 e^{-r^2}\ r\ dr\ d\theta = 2\pi\int_0^2 e^{-r^2}\ r\ dr = 2\pi\left[\frac{e^{-r^2}}{-2}\right]_0^2 = 2\pi\left(\frac{e^{-4}}{-2}-\frac{e^0}{-2}\right)$$
$$= \pi(-e^{-4}+1) = \pi(1-e^{-4})$$

7. (5 points) Find the area of one loop of the rose $r = \sin(3\theta)$.

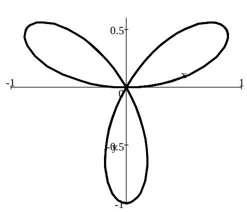
a. $\frac{\pi}{12}+\frac{\sqrt{3}}{48}$

b. $\frac{\pi}{12}-\frac{\sqrt{3}}{48}$

c. $\frac{\pi}{12}+\frac{1}{24}$

d. $\frac{\pi}{12}-\frac{1}{24}$

e. $\frac{\pi}{12}$ correctchoice



The first loop ends when $3\theta = \pi$ or $\theta = \frac{\pi}{3}$.

$$A = \int_0^{\pi/3}\int_0^{\sin(3\theta)} 1 \bullet r\ dr\ d\theta = \int_0^{\pi/3}\left[\frac{r^2}{2}\right]_0^{\sin(3\theta)} d\theta = \int_0^{\pi/3}\frac{\sin^2(3\theta)}{2}\ d\theta$$
$$= \int_0^{\pi/3}\frac{1-\cos(6\theta)}{4}\ d\theta = \frac{1}{4}\left[\theta-\frac{\sin(6\theta)}{6}\right]_0^{\pi/3} = \frac{\pi}{12}$$

8. (5 points) Find the mass of the cylinder $x^2+y^2 \le 4$ for $0 \le z \le 3$ if the density is $\rho = x^2+y^2+z^2$.

a. 24π

b. 30π

c. 36π

d. 52π

e. 60π correctchoice

$$M = \iiint \rho\ dV = \int_0^3\int_0^{2\pi}\int_0^2 (r^2+z^2)\ r\ dr\ d\theta\ dz = 2\pi\int_0^3\int_0^2 (r^3+rz^2)\ dr\ dz$$
$$= 2\pi\int_0^3\left[\frac{r^4}{4}+\frac{r^2z^2}{2}\right]_0^2 dz = 2\pi\int_0^3 (4+2z^2)\ dz = 2\pi\left[4z+\frac{2z^3}{3}\right]_0^3 = 2\pi[12+18] = 60\pi$$

9. (20 points) Find the mass M and center of mass $(\bar{x}, \bar{y})$ of the quarter of the circle $x^2 + y^2 \leq 4$ in the first quadrant if the density is $\rho = 3 + x^2 + y^2$.
HINT: By symmetry, $\bar{x} = \bar{y}$. So you only need to compute $\bar{x}$.

$$M = \iint \rho \ dA = \int_0^{\pi/2} \int_0^2 (3 + r^2) \ r \ dr \ d\theta = \frac{\pi}{2} \int_0^2 (3r + r^3) \ dr = \frac{\pi}{2} \left[\frac{3r^2}{2} + \frac{r^4}{4} \right]_0^2$$

$$= \frac{\pi}{2}(6 + 4) = 5\pi$$

$$x\text{-mom} = M_y = \iint x\rho \ dA = \int_0^{\pi/2} \int_0^2 (r\cos(\theta))(3 + r^2) \ r \ dr \ d\theta$$

$$= \int_0^{\pi/2} \cos(\theta) \ d\theta \ \int_0^2 (3r^2 + r^4) \ dr = [\sin(\theta)]_{\theta=0}^{\pi/2} \left[r^3 + \frac{r^5}{5} \right]_{r=0}^2 = 1 \bullet \left(8 + \frac{32}{5} \right) = \frac{72}{5}$$

$$\bar{x} = \frac{x - \text{mom}}{M} = \frac{M_y}{M} = \frac{72}{5 \bullet 5\pi} = \frac{72}{25\pi} = \bar{y}.$$

11. (20 points) Find the volume V and the z-component of the centroid $\bar{z}$ of the hemisphere $0 \leq z \leq \sqrt{9 - x^2 - y^2}$.

The volume of a hemisphere of radius 3 is simply $V = \frac{1}{2} \bullet \frac{4}{3}\pi R^3 = \frac{2}{3}\pi(3)^3 = 18\pi$, but it may be computed from the integral

$$V = \iiint 1 \ dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^3 \rho^2 \sin\varphi \ d\rho \ d\theta \ d\varphi = \left[\frac{\rho^3}{3} \right]_0^3 2\pi \left[-\cos\varphi \right]_0^{\pi/2}$$

$$= [9] 2\pi \left[-\cos\frac{\pi}{2} + \cos 0 \right] = 18\pi$$

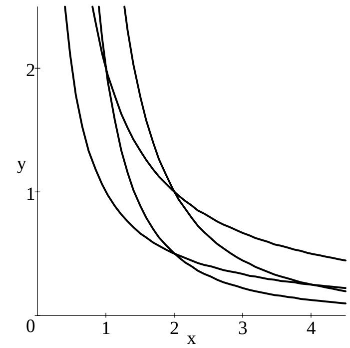
$$z\text{-mom} = M_{xy} = \iiint z \ dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^3 \rho\cos\varphi \ \rho^2 \sin\varphi \ d\rho \ d\theta \ d\varphi = \left[\frac{\rho^4}{4} \right]_0^3 2\pi \left[\frac{\sin^2\varphi}{2} \right]_0^{\pi/2}$$

$$= \frac{81}{4} 2\pi \frac{1}{2} = \frac{81\pi}{4}$$

$$\bar{z} = \frac{z - \text{mom}}{M} = \frac{M_{xy}}{M} = \frac{81\pi}{4} \bullet \frac{1}{18\pi} = \frac{9}{8}$$

10. (20 points) Compute $\iint_R x^2y\ dx\,dy$ over the diamond shaped region R bounded by

$$y = \frac{1}{x}, \qquad y = \frac{2}{x}, \qquad y = \frac{2}{x^2}, \qquad y = \frac{4}{x^2}$$



FULL CREDIT for integrating in the curvilinear coordinates $u = xy$ and $v = x^2y$. (Solve for x and y.)

HALF CREDIT for integrating in rectangular coordinates.

METHOD 1: Integrand: $x^2y = v$ Limits: $1 \le u \le 2$ $\quad 2 \le v \le 4$

Solve for x and y : $\frac{v}{u} = \frac{x^2y}{xy} = x \qquad y = \frac{u}{x} = u\frac{u}{v} = \frac{u^2}{v}$

$R(u,v) = \left(u^{-1}v,\ u^2v^{-1}\right) \qquad R_u = \left(-u^{-2}v,\ 2uv^{-1}\right) \qquad R_v = \left(u^{-1},\ -u^2v^{-2}\right)$

$$J = \left|\begin{vmatrix} -u^{-2}v & u^{-1} \\ 2uv^{-1} & -u^2v^{-2} \end{vmatrix}\right| = |u^{-2}vu^2v^{-2} - u^{-1}2uv^{-1}| = \left|\frac{1}{v} - \frac{2}{v}\right| = \left|-\frac{1}{v}\right| = \frac{1}{v}$$

$$\iint_R x^2y\ dx\ dy = \int_2^4\int_1^2 v\ \frac{1}{v}\ du\ dv = \int_2^4\int_1^2 1\ du\ dv = (2-1)(4-2) = 2$$

METHOD 2: Find intersections:

$$\frac{2}{x} = \frac{2}{x^2} \Rightarrow 2x^2 = 2x \Rightarrow x = 1, \qquad \frac{1}{x} = \frac{2}{x^2} \Rightarrow x^2 = 2x \Rightarrow x = 2$$

$$\frac{2}{x} = \frac{4}{x^2} \Rightarrow 2x^2 = 4x \Rightarrow x = 2, \qquad \frac{1}{x} = \frac{4}{x^2} \Rightarrow x^2 = 4x \Rightarrow x = 4$$

So the integral breaks into two pieces:

$$\iint_R x^2y\ dx\,dy = \int_1^2\int_{2/x^2}^{2/x} x^2y\ dy\ dx + \int_2^4\int_{1/x}^{4/x^2} x^2y\ dy\ dx = \int_1^2\left[x^2\frac{y^2}{2}\right]_{y=2/x^2}^{2/x} dx + \int_2^4\left[x^2\frac{y^2}{2}\right]_{y=1/x}^{4/x^2}$$

$$= \frac{1}{2}\int_1^2\left[x^2\frac{4}{x^2}\right] - \left[x^2\frac{4}{x^4}\right] dx + \frac{1}{2}\int_2^4\left[x^2\frac{16}{x^4}\right] - \left[x^2\frac{1}{x^2}\right] dx$$

$$= \frac{1}{2}\int_1^2\left(4 - \frac{4}{x^2}\right) dx + \frac{1}{2}\int_2^4\left(\frac{16}{x^2} - 1\right) dx = \frac{1}{2}\left[4x + \frac{4}{x}\right]_1^2 + \frac{1}{2}\left[-\frac{16}{x} - x\right]_2^4$$

$$= \frac{1}{2}([8+2] - [4+4] + [-4-4] - [-8-2]) = \frac{1}{2}(10 - 8 - 8 + 10) = 2$$