MATH 253 FINAL EXAM Spring 1998

Sections 501-503 Solutions P. Yasskin

Part I: Multiple Choice (5 points each) No Partial Credit

1. $\lim_{n\to\infty} \dfrac{n^2}{n^2+(-1)^n n} =$

a. 0

b. 1 correctchoice

c. 2

d. 4

e. divergent

$$\lim_{n\to\infty} \frac{n^2}{n^2+(-1)^n n} = \lim_{n\to\infty} \frac{1}{1+\frac{(-1)^n}{n}} = 1$$

2. The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is

a. absolutely convergent

b. conditionally convergent correctchoice

c. divergent

d. none of these

$\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is an alternating, decreasing series and $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+1} = 0.$

So it is convergent by the Alternating Series Test.

Further, the related absolute series is $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ which is divergent by the Integral Test, since $\displaystyle\int_1^{\infty} \frac{n}{n^2+1}\,dn = \left[\frac{1}{2}\ln(n^2+1)\right]_1^{\infty} = \infty.$

Thus the original series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is conditionally convergent.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+1} =$

a. 0

b. $\frac{1}{2}$

c. 1

d. 2

e. divergent correctchoice

Since $\displaystyle\lim_{n=1} \frac{n^2}{n^2+1} = 1 \ne 0,$ the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$ is divergent by the nth Term Divergence Test.

4. The series $\sum_{n=1}^{\infty} \frac{n}{n^{1.5}+1}$ is

a. convergent by the Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^{.5}}$.

b. conv. by the Limit Comp. Test with $\sum_{n=1}^{\infty} \frac{1}{n^{.5}}$ but not by the Comp. Test.

c. divergent by the Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^{.5}}$.

d. div. by the Limit Comp. Test with $\sum_{n=1}^{\infty} \frac{1}{n^{.5}}$ but not by the Comp. Test.

correctchoice

e. none of these

For large n, the term $a_n = \frac{n}{n^{1.5}+1}$ is approximately like $b_n = \frac{1}{n^{.5}}$ so we want to compare to $\sum_{n=1}^{\infty} \frac{1}{n^{.5}}$ which is a divergent p-series since $p = .5 < 1$. Since $\frac{n}{n^{1.5}+1} < \frac{n}{n^{1.5}} = \frac{1}{n^{.5}}$, the Comparison Test does not apply. So we try the Limit Comparison Test: Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^{1.5}+1} \bullet \frac{n^{.5}}{1} = \lim_{n\to\infty} \frac{n^{1.5}}{n^{1.5}+1} = 1$, the series $\sum_{n=1}^{\infty} \frac{n}{n^{1.5}+1}$ also diverges.

5. $\sum_{n=1}^{\infty} \frac{2}{4n^2-1} =$ (Note: $\frac{2}{4n^2-1} = \frac{1}{2n-1} - \frac{1}{2n+1}$)

a. 0

b. $\frac{1}{2}$

c. 1 correctchoice

d. 2

e. divergent

We first compute the partial sum:

$$S_k = \sum_{n=1}^{k} \frac{2}{4n^2-1} = \sum_{n=1}^{k} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

$$= \left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \cdots + \left(\frac{1}{2k-3} - \frac{1}{2k-1} \right) + \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) = 1 - \frac{1}{2k+1}$$

So the infinite sum is $\sum_{n=1}^{\infty} \frac{2}{4n^2-1} = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \left(1 - \frac{1}{2k+1} \right) = 1$

6. Consider the Taylor series about $x = 0$ for $f(x) = e^{-x}$. What is the minimum degree of the Taylor polynomial you should use to approximate $e^{-0.1}$ to within $\pm 10^{-8}$? Give the degree n of the highest power of x that you need to ***keep***.

a. 1

b. 3

c. 5 correctchoice

d. 7

e. 9

Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$

we have $e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$ and

$$e^{-0.1} = \sum_{n=0}^{\infty} (-1)^n \frac{(0.1)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{10^{-n}}{n!} = 1 - 10^{-1} + \frac{10^{-2}}{2} - \frac{10^{-3}}{6} + \frac{10^{-4}}{24} - \frac{10^{-5}}{120} + \frac{10^{-6}}{720} +$$

Since this is an alternating decreasing series, the error is bounded by the next term.

Since $\frac{10^{-5}}{120} > 10^{-8} > \frac{10^{-6}}{720}$, we must keep up to the $n = 5$ term.

7. Find the volume of the solid under the plane $z = x$ and above the triangle with vertices $(1,1)$, $(2,1)$ and $(1,4)$.

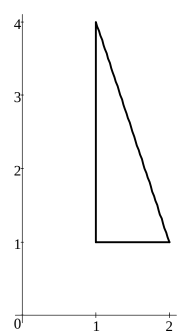
a. 1

b. 2 correctchoice

c. 3

d. 4

e. $\frac{9}{2}$



The base is shown. The top is $z = x$.

The diagonal side of the base has slope $m = -3$.

So its equation is $y - 1 = -3(x - 2)$ or $y = -3x + 7$.

$$V = \int_1^2 \int_1^{-3x+7} x \, dy \, dx = \int_1^2 \Big[xy \Big]_{y=1}^{-3x+7} dx = \int_1^2 [x(-3x+7)] - [x] \, dx$$

$$= \int_1^2 -3x^2 + 6x \, dx = \Big[-x^3 + 3x^2 \Big]_{x=1}^{2} = [-8 + 12] - [-1 + 3] = 2$$

8. A 5 lb mass moves up the helix $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$ for $0 \le t \le \pi$. Find the work done against the force of gravity $\vec{F} = -5\hat{k}$.

- **a.** -4π
- **b.** -5π
- **c.** -20π correctchoice
- **d.** -80π
- **e.** -100π

$\vec{v} = (-3\sin t, 3\cos t, 4)$ $\vec{F} = (0,0,-5)$

$W = \int_0^\pi \vec{F} \bullet \vec{v}\ dt = \int_0^\pi -20\ dt = -20\pi$

9. Compute the line integral $\int_C \vec{F} \bullet d\vec{s}$ counterclockwise around the circle $x^2 + y^2 = 4$ for the vector field $\vec{F} = (-y(x^2+y^2), x(x^2+y^2))$.

- **a.** 2π
- **b.** 4π
- **c.** 8π
- **d.** 16π
- **e.** 32π correctchoice

$\int_C \vec{F} \bullet d\vec{s} = \int_C P\ dx + Q\ dy$ where $P = -y(x^2+y^2) = -yx^2 - y^3$ and $Q = x(x^2+y^2) = x^3 + xy^2$.

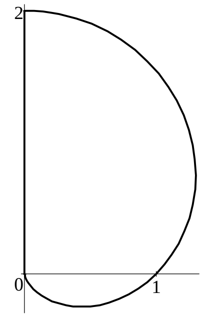
Let D denote the interior of the circle. Then, by Green's Theorem,

$$\int_C \vec{F} \bullet d\vec{s} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_D (3x^2+y^2) - (-x^2-3y^2)\,dx\,dy = \iint_D (4x^2+4y^2)\,dx\,dy$$

Switch to polar coordinates. $x^2 + y^2 = r^2$ $dx\,dy = r\,dr\,d\theta$ So:

$$\int_C \vec{F} \bullet d\vec{s} = \int_0^{2\pi} \int_0^2 4r^2\,r\,dr\,d\theta = 2\pi[r^4]_0^2 = 32\pi$$

10. Find the total mass of a plate bounded by the right half of the cardioid $r = 1 + \sin\theta$ and the y-axis if the mass density is $\rho = 3x$.



a. 4 correctchoice

b. π

c. 2

d. $\frac{\pi}{2}$

e. $\frac{1}{2}$

$$M = \int\int \rho \; dA = \int_{-\pi/2}^{\pi/2}\int_0^{1+\sin\theta} 3(r\cos\theta)\, r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \cos\theta\, [r^3]_{r=0}^{1+\sin\theta}\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cos\theta\,(1+\sin\theta)^3\, d\theta = \frac{(1+\sin\theta)^4}{4}\Bigg|_{-\pi/2}^{\pi/2} = \left[\frac{2^4}{4}\right] - \left[\frac{0}{4}\right] = 4$$

11. Find the area of the piece of the paraboloid $z = 9 - x^2 - y^2$ in the first octant.

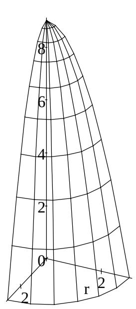
a. $\frac{\pi}{16}\left[(37)^{3/2} - 1\right]$

b. $\frac{\pi}{24}\left[(37)^{3/2} - 1\right]$ correct

c. $\frac{\pi}{16}(37)^{3/2}$

d. $\frac{\pi}{4}(37)^{3/2}$

e. $\frac{9\pi}{4}$



The paraboloid may be parametrized as

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 9 - r^2)$

The tangent and normal vectors are

$\vec{R}_r = (\cos\theta,\ \sin\theta, -2r)$

$\vec{R}_\theta = (-r\sin\theta, r\cos\theta,\ 0)$

$\vec{N} = \vec{R}_r \times \vec{R}_\theta = (2r^2\cos\theta, 2r^2\sin\theta, r)$

The length of the normal is

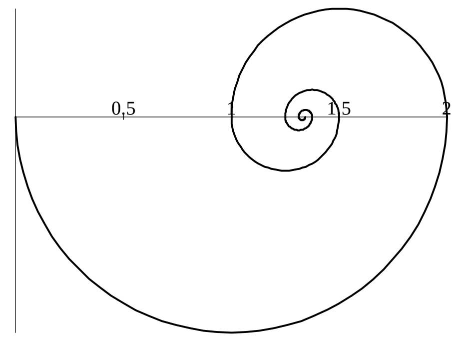
$\left|\vec{N}\right| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$

So the area is $A = \iint \left|\vec{N}\right| dr\, d\theta = \int_0^{\pi/2}\int_0^3 r\sqrt{4r^2+1}\; dr\, d\theta$

$$= \frac{\pi}{2}\left[\frac{2}{3} \bullet \frac{1}{8}(4r^2+1)^{3/2}\right]_0^3 = \frac{\pi}{24}\left[(37)^{3/2} - 1\right]$$

Part II: Work Out Problems Partial credit will be given.

12. (10 points) The spiral at the right is made from an infinite number of semicircles whose centers are all on the x-axis. The radius of each semicircle is half of the radius of the previous semicircle.



a. Consider the infinite sequence of points where the spiral crosses the x-axis. What is the x-coordinate of the limit of this sequence?

$$2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots = \sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n = \frac{2}{1 + \frac{1}{2}} = \frac{4}{3}$$

b. What is the total length of the spiral (with an infinite number of semicircles)? Or, is the length infinite?

Each semicircle has length $L_n = \pi r_n$ where the radii are $r_n = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$.

So the total length is $L = \sum_{n=0}^{\infty} \pi r_n = \sum_{n=0}^{\infty} \pi \left(\frac{1}{2}\right)^n = \frac{\pi}{1 - \frac{1}{2}} = 2\pi.$

13. (15 points) Find the interval of convergence for the series $\sum_{n=2}^{\infty} \frac{(x-3)^n}{2^n\ n\ \ln n}$

a. (2 pts) The center of convergence is $c =$ ___3___.

b. (7 pts) Find the radius of convergence. (Name the test you use.)

We apply the ratio test: $a_n = \frac{(x-3)^n}{2^n\ n\ \ln n}$ $\quad a_{n+1} = \frac{(x-3)^{n+1}}{2^{n+1}\ (n+1)\ \ln(n+1)}$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{2^{n+1}\ (n+1)\ \ln(n+1)}\ \frac{2^n\ n\ \ln n}{(x-3)^n}\right|$$

$$= \frac{|x-3|}{2}\lim_{n\to\infty}\frac{n}{n+1}\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = \frac{|x-3|}{2}\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}}$$

$$= \frac{|x-3|}{2}(1)(1) = \frac{|x-3|}{2}$$

The series converges if $\rho = \frac{|x-3|}{2} < 1$ or $|x-3| < 2$

$R =$ ___2___

c. (2 pts) Check the left endpoint. (Name the test you use.)

$x = 3-2 = 1$ The series becomes $\sum_{n=2}^{\infty}\frac{(-2)^n}{2^n\ n\ \ln n} = \sum_{n=2}^{\infty}\frac{(-1)^n}{n\ \ln n}$.

This is an alternating, decreasing series and $\lim_{n\to\infty}\frac{1}{n\ \ln n} = 0$.

So the series converges by the Alternating Series Test.

Circle: **[convergent]** / divergent

d. (2 pts) Check the right endpoint. (Name the test you use.)

$x = 3+2 = 5$ The series becomes $\sum_{n=2}^{\infty}\frac{(2)^n}{2^n\ n\ \ln n} = \sum_{n=2}^{\infty}\frac{1}{n\ \ln n}$.

$\int_2^{\infty}\frac{1}{n\ \ln n}\,dn = \int_{\ln 2}^{\infty}\frac{1}{u}\,du = [\ln u]_{\ln 2}^{\infty} = \infty.$ $\qquad u = \ln n \qquad du = \frac{1}{n}\,dn$

So the series diverges by the Integral Test.

Circle: convergent / **[divergent]**

e. (2 pts) The interval of convergence is ___$[1,5)$ or $1 \le x < 5$___.

14. (10 points) Let V be the solid hemisphere $x^2 + y^2 + z^2 \le 4$ for $z \ge 0$.
Let H be the hemisphere surface $x^2 + y^2 + z^2 = 4$ for $z \ge 0$.
Let D be the disk $x^2 + y^2 \le 4$ with $z = 0$.

Notice that H and D form the boundaryof V with outward normal provided H is oriented upward and D is oriented downward. Then Gauss' Theorem states

$$\iiint_V \vec{\nabla} \bullet \vec{F}\, dV = \iint_H \vec{F} \bullet d\vec{S} + \iint_D \vec{F} \bullet d\vec{S}$$

Compute $\iint_H \vec{F} \bullet d\vec{S}$ for $\vec{F} = \left(x^3 + y^2 + z^2,\ y^3 + x^2 + z^2,\ z^3 + x^2 + y^2\right)$ using one of the following methods: (Circle the method you choose.)

- Method I: Parametrize H and compute $\iint_H \vec{F} \bullet d\vec{S}$ explicitly.
- Method II: Parametrize D, compute $\iint_D \vec{F} \bullet d\vec{S}$ and $\iiint_V \vec{\nabla} \bullet \vec{F}\, dV$ and solve for $\iint_H \vec{F} \bullet d\vec{S}$.

Method I: (You don't want to do it this way!) Parametrize H:

$$\vec{R}(\varphi,\theta) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi) \qquad 0 \le \varphi \le \frac{\pi}{2} \qquad 0 \le \theta \le 2\pi$$

$$\vec{R}_\varphi = (2\cos\varphi\cos\theta, 2\cos\varphi\sin\theta, -2\sin\varphi)$$

$$\vec{R}_\theta = \left(-2\sin\varphi\sin\theta,\ 2\sin\varphi\cos\theta,\ 0\right)$$

$$\vec{N} = \left(4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi\right)$$

$$\vec{F} = \left(x^3 + y^2 + z^2,\ y^3 + x^2 + z^2,\ z^3 + x^2 + y^2\right)$$
$$= (8\sin^3\varphi\cos^3\theta + 4\sin^2\varphi\sin^2\theta + 4\cos^2\varphi,\ 8\sin^3\varphi\sin^3\theta + 4\sin^2\varphi\cos^2\theta + 4\cos^2\varphi,$$
$$8\cos^3\varphi + 4\sin^2\varphi\cos^2\theta + 4\sin^2\varphi\sin^2\theta)$$

$$\vec{F} \bullet \vec{N} = \left(8\sin^3\varphi\cos^3\theta + 4\sin^2\varphi\sin^2\theta + 4\cos^2\varphi\right)\left(4\sin^2\varphi\cos\theta\right) + \left(8\sin^3\varphi\sin^3\theta + 4\sin^2\varphi\cos^2\theta\right.$$
$$\left. + \left(8\cos^3\varphi + 4\sin^2\varphi\right)(4\sin\varphi\cos\varphi)\right.$$
$$= 16[2\sin^5\varphi\cos^4\theta + \sin^4\varphi\sin^2\theta\cos\theta + \cos^2\varphi\sin^2\varphi\cos\theta + 2\sin^5\varphi\sin^4\theta + \sin^4\varphi\cos^2\theta\sin\theta$$
$$+ \cos^2\varphi\sin^2\varphi\sin\theta + 2\cos^4\varphi\sin\varphi + \sin^3\varphi\cos\varphi]$$
$$= 16[2\sin^5\varphi\cos^4\theta + \sin^4\varphi\sin^2\theta\cos\theta + \cos^2\varphi\sin^2\varphi\cos\theta + 2\sin^5\varphi\sin^4\theta + \sin^4\varphi\cos^2\theta\sin\theta$$
$$+ \cos^2\varphi\sin^2\varphi\sin\theta + 2\cos^4\varphi\sin\varphi + \sin^3\varphi\cos\varphi]$$

$$\iint_H \vec{F} \bullet d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} \vec{F} \bullet \vec{N}\, d\varphi\, d\theta$$
$$= 16\int_0^{2\pi}\int_0^{\pi/2} [2\sin^5\varphi\cos^4\theta + 2\sin^5\varphi\sin^4\theta + 2\cos^4\varphi\sin\varphi + \sin^3\varphi\cos\varphi]\, d\varphi\, d\theta$$

since

$$\int_0^{2\pi} \sin\theta\, d\theta = 0 \qquad \int_0^{2\pi} \cos\theta\, d\theta = 0 \qquad \int_0^{2\pi} \sin^2\theta\cos\theta\, d\theta = 0 \qquad \int_0^{2\pi} \cos^2\theta\sin\theta\, d\theta = 0$$

Further, after much work:

$$\int_0^{\pi/2} \sin^5\varphi\, d\varphi = \frac{8}{15} \qquad \int_0^{2\pi} \cos^4\theta\, d\theta = \int_0^{2\pi} \sin^4\theta\, d\theta = \frac{3}{4}\pi$$

$$\int_0^{\pi/2} \cos^4\varphi\sin\varphi\, d\varphi = \frac{1}{5} \qquad \int_0^{\pi/2} \sin^3\varphi\cos\varphi\, d\varphi = \frac{1}{4} \qquad \text{and} \qquad \int_0^{2\pi} 1\, d\theta = 2\pi$$

So

$$\int_0^{2\pi}\int_0^{\pi/2} \vec{F} \bullet \vec{N}\ d\varphi\ d\theta = 16\left[2\left(\frac{8}{15}\right)\left(\frac{3}{4}\pi\right) + 2\left(\frac{8}{15}\right)\left(\frac{3}{4}\pi\right) + 2\left(\frac{1}{5}\right)(2\pi) + \frac{1}{4}(2\pi)\right] = \frac{232}{5}\pi$$

Method II: Parametrize D:

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 0) \qquad 0 \le r \le 2 \qquad 0 \le \theta \le 2\pi$

$\vec{R}_r = \left(\ \cos\theta,\ \sin\theta,\ 0\right)$

$\vec{R}_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\vec{N} = \left(\ 0,\ 0,\ r\right)$ This is up but we need down; so we reverse it to get $\vec{N} = (0,0,-r)$

$\vec{F} = \left(x^3 + y^2 + z^2,\ y^3 + x^2 + z^2,\ z^3 + x^2 + y^2\right) = \left(r^3\cos^3\theta + r^2\sin^2\theta, r^3\sin^3\theta + r^2\cos^2\theta, r^2\right)$

$\vec{F} \bullet \vec{N} = -r^3$

$$\iint_D \vec{F} \bullet d\vec{S} = \int_0^{2\pi}\int_0^2 \vec{F} \bullet \vec{N}\ dr\ d\theta = \int_0^{2\pi}\int_0^2 -r^3\ dr\ d\theta = -2\pi\left[\frac{r^4}{4}\right]_0^2 = -8\pi$$

Use spherical coordinates for V:

$\vec{R}(\rho,\varphi,\theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi) \qquad 0 \le \rho \le 2, \qquad 0 \le \varphi \le \frac{\pi}{2} \qquad 0 \le \theta \le 2\pi$

$dV = \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta$

$\vec{\nabla} \bullet \vec{F} = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$

$$\iiint_V \vec{\nabla} \bullet \vec{F}\ dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 3\rho^2\ \rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta = 2\pi[-\cos\varphi]_{\varphi=0}^{\pi/2}\left[\frac{3\rho^5}{5}\right]_{\rho=0}^2$$

$$= 2\pi[1]\left[\frac{3 \bullet 32}{5}\right] = \frac{192}{5}\pi$$

We combine these to get

$$\iint_H \vec{F} \bullet d\vec{S} = \iiint_V \vec{\nabla} \bullet \vec{F}\ dV - \iint_D \vec{F} \bullet d\vec{S} = \frac{192}{5}\pi - -8\pi = \frac{232}{5}\pi$$

Notice that Methods I and II give the same answer.

15. (10 points) Find the point (x, y, z) in the first octant on the surface $z = \frac{27}{x} + \frac{64}{y}$ which is closest to the origin.

Minimize the square of the distance to the origin $f = x^2 + y^2 + z^2$ subject to the constraint that the point lies on the surface $z = \frac{27}{x} + \frac{64}{y}$.

Method I: Eliminate a constraint:

Minimize $f = x^2 + y^2 + \left(\frac{27}{x} + \frac{64}{y}\right)^2$

$$\vec{\nabla} f = \left(2x + 2\left(\frac{27}{x} + \frac{64}{y}\right)\left(\frac{-27}{x^2}\right), 2y + 2\left(\frac{27}{x} + \frac{64}{y}\right)\left(\frac{-64}{y^2}\right)\right) = (0, 0)$$

Multiply the first equation by $\frac{x^2}{54}$ and the second equation by $\frac{y^2}{128}$:

(1.) $\frac{x^3}{27} = \left(\frac{27}{x} + \frac{64}{y}\right)$ (2.) $\frac{y^3}{64} = \left(\frac{27}{x} + \frac{64}{y}\right)$

Equate these to obtain $\frac{x}{3} = \frac{y}{4}$ and plug back into (1.) to obtain:

$$\frac{x^3}{27} = \left(\frac{27}{x} + \frac{16 \bullet 3}{x}\right) = \frac{75}{x}$$

Cross multiply: $x^4 = 75 \bullet 27 = 3^4 \bullet 5^2$ So $x = 3\sqrt{5}$ and $y = 4\sqrt{5}$

and $z = \frac{27}{x} + \frac{64}{y} = \frac{27}{3\sqrt{5}} + \frac{64}{4\sqrt{5}} = \frac{25}{\sqrt{5}} = 5\sqrt{5}$.

Method II: Lagrange multipliers:

Minimize $f = x^2 + y^2 + z^2$ subject to $g = z - \frac{27}{x} - \frac{64}{y} = 0$

$\vec{\nabla} f = (2x, 2y, 2z)$ $\vec{\nabla} g = \left(\frac{27}{x^2}, \frac{64}{y^2}, 1\right)$

Lagrange equations: $\vec{\nabla} f = \lambda \vec{\nabla} g$:

$2x = \frac{27\lambda}{x^2}$ $2y = \frac{64\lambda}{y^2}$ $2z = \lambda$

Cross multiply in the first two equations and replace the λ by $2z$:

$2x^3 = 54z$ $2y^3 = 128z$

Divide by 2 and take the cube root: $x = 3\sqrt[3]{z}$ $y = 4\sqrt[3]{z}$

Plug into the constraint and solve: $z = \frac{27}{3\sqrt[3]{z}} + \frac{64}{4\sqrt[3]{z}} = \frac{25}{\sqrt[3]{z}}$ $z^{4/3} = 5^2$

$z = 5^{3/2} = 5\sqrt{5}$ $x = 3\sqrt[3]{z} = 3\sqrt{5}$ $y = 4\sqrt[3]{z} = 4\sqrt{5}$