Name________________________ ID____________ Section______

MATH 253 EXAM 2 Fall 1998

Sections 501-503 Solutions P. Yasskin

Multiple Choice: (5 points each)

1. Compute $\int_1^2 \int_1^x y\, dy\, dx$.

a. $-\frac{1}{3}$

b. $\frac{1}{3}$

c. $\frac{2}{3}$ correctchoice

d. $\frac{7}{6}$

e. $\frac{4}{3}$

$$\int_1^2 \int_1^x y\, dy\, dx = \int_1^2 \left[\frac{y^2}{2} \right]_{y=1}^{x} dx = \int_1^2 \frac{x^2}{2} - \frac{1}{2}\, dx = \left[\frac{x^3}{6} - \frac{x}{2} \right]_1^2$$

$$= \left[\frac{8}{6} - \frac{2}{2} \right] - \left[\frac{1}{6} - \frac{1}{2} \right] = \frac{2}{3}$$

2. Find the volume under the surface $z = 2x^2 y$ above the triangle with vertices $(0,0)$, $(1,2)$ and $(0,4)$.

a. $-\frac{1}{3}$

b. $\frac{1}{3}$

c. $\frac{2}{3}$

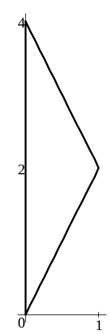
d. $\frac{7}{6}$

e. $\frac{4}{3}$ correctchoice

$$\int_0^1 \int_{2x}^{4-2x} 2x^2 y\, dy\, dx = \int_0^1 \left[x^2 y^2 \right]_{y=2x}^{4-2x} dx = \int_0^1 x^2 \left[(4-2x)^2 - (2x)^2 \right] dx$$

$$= \int_0^1 x^2 (16 - 16x)\, dx = \int_0^1 16x^2 - 16x^3\, dx = \left[\frac{16}{3} x^3 - 4x^4 \right]_0^1$$

$$= \left[\frac{16}{3} - 4 \right] = \frac{4}{3}$$



3. Compute $\int_0^1 \int_{\sqrt{y}}^1 \int_0^y x\ y\ dz\ dx\ dy$.

a. $\dfrac{1}{24}$ correctchoice

b. $\dfrac{1}{12}$

c. $\dfrac{1}{2\sqrt{2}}$

d. $\dfrac{3}{2\sqrt{2}}$

e. $3\sqrt{2}$

$$\int_0^1 \int_{\sqrt{y}}^1 \int_0^y x\ y\ dz\ dx\ dy = \int_0^1 \int_{\sqrt{y}}^1 x\ y\ z\ \Big|_{z=0}^{y}\ dx\ dy = \int_0^1 \int_{\sqrt{y}}^1 x\ y^2\ dx\ dy$$

$$= \int_0^1 \left[\frac{x^2\ y^2}{2} \right]_{x=\sqrt{y}}^{1} dy = \int_0^1 \frac{y^2}{2} - \frac{y^3}{2}\ dy = \left[\frac{y^3}{6} - \frac{y^4}{8} \right]_{y=0}^{1} = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$$

4. Find the area enclosed by one loop of the daisy $r = \sin 4\theta$:

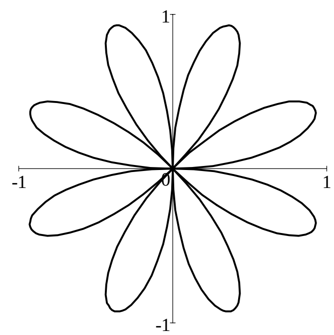
a. $\dfrac{\pi}{32}$

b. $\dfrac{\pi}{16}$ correctchoice

c. $\dfrac{\pi}{8}$

d. $\dfrac{\pi}{4}$

e. $\dfrac{\pi}{2}$



$$A = \iint 1\ dA = \int_0^{\pi/4} \int_0^{\sin 4\theta} r\ dr\ d\theta = \int_0^{\pi/4} \frac{r^2}{2} \Big|_{r=0}^{\sin 4\theta} d\theta = \int_0^{\pi/4} \frac{\sin^2 4\theta}{2}\ d\theta$$

$$= \int_0^{\pi/4} \frac{1 - \cos 8\theta}{4}\ d\theta = \frac{1}{4}\left[\theta - \frac{\sin 8\theta}{8} \right]_{\theta=0}^{\pi/4} = \frac{1}{4}\,\frac{\pi}{4} = \frac{\pi}{16}$$

5. Compute $\displaystyle\int_0^{\sqrt[4]{\pi}} \int_{y^2}^{\sqrt{\pi}} y\sin(x^2)\ dx\ dy$.

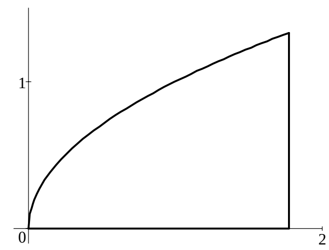
a. $\frac{1}{4}$

b. $\frac{1}{2}$ correctchoice

c. $\frac{\pi}{4}$

d. $\frac{\pi}{2}$

e. $\frac{3}{4}\pi\sqrt{\pi}$



$$\int_0^{\sqrt[4]{\pi}} \int_{y^2}^{\sqrt{\pi}} y\sin(x^2)\ dx\ dy = \int_0^{\sqrt{\pi}} \int_0^{\sqrt{x}} y\sin(x^2)\ dy\ dx = \int_0^{\sqrt{\pi}} \left[\frac{y^2}{2}\sin(x^2) \right]_{y=0}^{\sqrt{x}} dx \qquad u = x^2$$

$$= \int_0^{\sqrt{\pi}} \frac{x}{2}\sin(x^2)\ dx = \int_{x=0}^{\sqrt{\pi}} \frac{1}{4}\sin u\ du = \left[-\frac{1}{4}\cos u \right]_{x=0}^{\sqrt{\pi}} \qquad du = 2x\ dx$$

$$= \left[-\frac{1}{4}\cos x^2 \right]_{x=0}^{\sqrt{\pi}} = -\frac{1}{4}\cos\pi + \frac{1}{4}\cos 0 = \frac{1}{2}$$

6. Compute $\displaystyle\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\ dx\ dy$ by converting to polar coordinates.

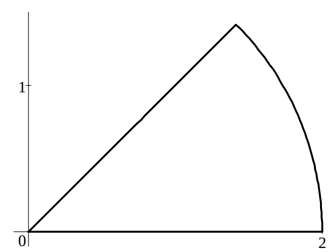
a. $\frac{\pi}{4}\ln 3$

b. $\frac{\pi}{8}\ln 3$

c. $\frac{\pi}{4}\ln 5$

d. $\frac{\pi}{8}\ln 5$ correctchoice

e. $\frac{\pi}{4}\ln 17$



$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\ dx\ dy = \int_0^{\pi/4} \int_0^2 \frac{1}{1+r^2}\ r\ dr\ d\theta = \frac{\pi}{4}\left[\frac{1}{2}\ln(1+r^2) \right]_{r=0}^2 = \frac{\pi}{8}\ln 5$$

7. Compute $\iiint_D \sqrt{x^2+y^2}\ dV$ over the region D bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane.

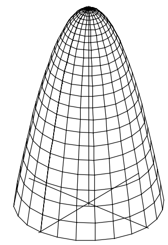
a. $\frac{4\pi}{5}3^4$ correctchoice

b. $\frac{\pi}{2}3^4$

c. $\frac{\pi}{2}3^5$

d. $2\pi 3^4$

e. $2\pi 3^5$



$$\iiint_D \sqrt{x^2+y^2}\ dV = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r\ r\ dz\ dr\ d\theta = 2\pi\int_0^3\int_0^{9-r^2} r^2\ dz\ dr = 2\pi\int_0^3\Big[r^2 z\Big]_{z=0}^{9-r^2} dr$$

$$= 2\pi\int_0^3 r^2(9-r^2)\ dr = 2\pi\left[\frac{9r^3}{3} - \frac{r^5}{5}\right]_0^3 = 2\pi\left[\frac{9\bullet 3^3}{3} - \frac{3^5}{5}\right] = 2\pi 3^4\left[1 - \frac{3}{5}\right] = \frac{4\pi}{5}3^4$$

8. Compute $\iiint_H z\ dV$ over the solid hemisphere H below the sphere $x^2+y^2+z^2 = 4$ and above the xy-plane.

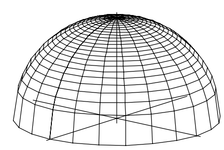
a. π

b. 2π

c. 4π correctchoice

d. $\frac{4\pi}{3}$

e. $\frac{8\pi}{3}$

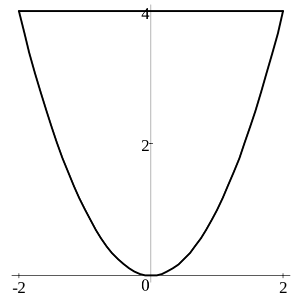


$$\iiint_H z\ dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho\cos\phi\ \rho^2\sin\phi\ d\rho\ d\phi\ d\theta = \left[\frac{\rho^4}{4}\right]_0^2\left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2}[2\pi]$$

$$= \left[\frac{16}{4}\right]_0^2\left[\frac{1}{2}\right]_0^{\pi/2}[2\pi] = 4\pi$$

9. (20 points) Find the mass M and center of mass $(\bar{x}, \bar{y})$ of the region above the parabola $y = x^2$ below the line $y = 4$, if the density is $\rho = y$. (12 points for setup.)

HINT: By symmetry, $\bar{x} = 0$. So you only need to compute $\bar{y}$.



$$M = \int\int \rho \; dA = \int_{-2}^{2}\int_{x^2}^{4} y \; dy \; dx = \int_{-2}^{2}\left[\frac{y^2}{2}\right]_{y=x^2}^{4} dx = \int_{-2}^{2} 8 - \frac{x^4}{2} \; dx = \left[8x - \frac{x^5}{10}\right]_{-2}^{2}$$

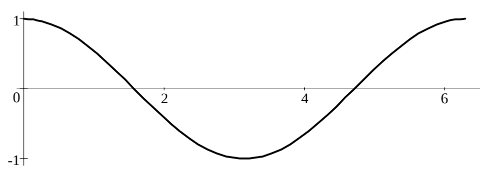
$$= 2\left[16 - \frac{32}{10}\right] = 32\left[1 - \frac{1}{5}\right] = \frac{128}{5}$$

$$y\text{-mom} = \int\int y\rho \; dA = \int_{-2}^{2}\int_{x^2}^{4} y^2 \; dy \; dx = \int_{-2}^{2}\left[\frac{y^3}{3}\right]_{y=x^2}^{4} dx = \int_{-2}^{2} \frac{64}{3} - \frac{x^6}{3} \; dx$$

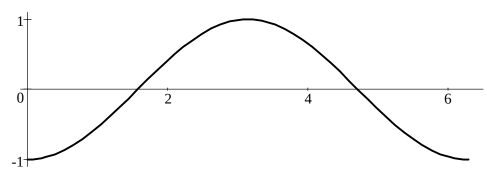
$$= \frac{1}{3}\left[64x - \frac{x^7}{7}\right]_{-2}^{2} = \frac{2}{3}\left[128 - \frac{128}{7}\right] = \frac{256}{3}\left(\frac{6}{7}\right) = \frac{512}{7}$$

$$\bar{y} = \frac{y\text{-mom}}{M} = \frac{512}{7}\frac{5}{128} = \frac{20}{7}$$

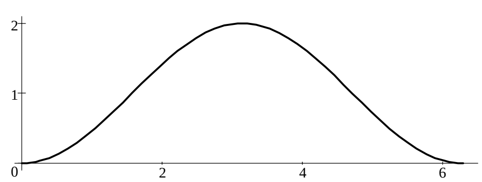
10. (10 points) Plot the polar curve $r = 1 - \cos\theta$. Show your work.



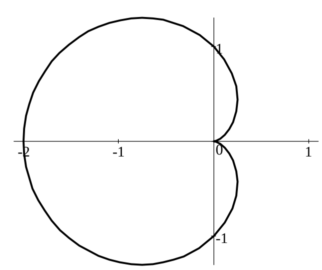
$\cos\theta$



$-\cos\theta$

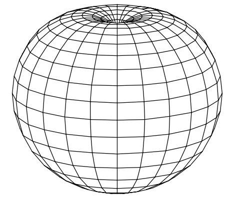


Rectangular: $r = 1 - \cos\theta$



Polar: $r = 1 - \cos\theta$

11. (20 points) Find the volume V and the z-component of the centroid $\bar{z}$ of the apple given in spherical coordinates by $\rho = 1 - \cos\phi$. (16 points for setup.)
Hint: In the ϕ-integral, use the substitution $u = 1 - \cos\phi$.



$$V = \iiint 1 \; dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta = 2\pi \int_0^{\pi}\left[\frac{\rho^3}{3}\right]_{\rho=0}^{1-\cos\phi} \sin\phi \; d\phi$$

$$= \frac{2\pi}{3}\int_0^{\pi}(1-\cos\phi)^3 \sin\phi \; d\phi = \frac{2\pi}{3}\int u^3 \; du \qquad u = 1 - \cos\phi$$

$$= \frac{2\pi}{3}\left[\frac{u^4}{4}\right] \qquad = \frac{\pi}{6}\left[(1-\cos\phi)^4\right]_0^{\pi} = \frac{\pi}{6}(2)^4 = \frac{8\pi}{3} \qquad du = \sin\phi \; d\phi$$

z-mom$= \iiint z \; dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} \rho\cos\phi \; \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$

$$= 2\pi \int_0^{\pi}\left[\frac{\rho^4}{4}\right]_{\rho=0}^{1-\cos\phi} \cos\phi \; \sin\phi \; d\phi = \frac{\pi}{2}\int_0^{\pi}(1-\cos\phi)^4 \cos\phi \; \sin\phi \; d\phi \qquad u = 1 - \cos\phi$$

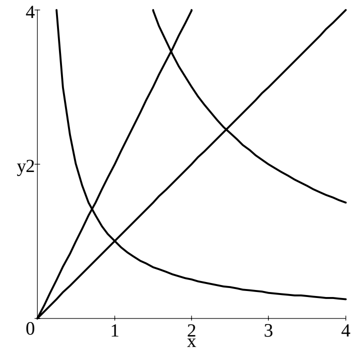
$$= \frac{\pi}{2}\int_0^{2} u^4(1-u) \; du = \frac{\pi}{2}\int_0^{2} u^4 - u^5 \; du = \frac{\pi}{2}\left[\frac{u^5}{5} - \frac{u^6}{6}\right]_{u=0}^{2} \qquad \cos\phi = 1 - u$$

$$= \frac{\pi}{2}\left[\frac{2^5}{5} - \frac{2^6}{6}\right] = 16\pi\left(\frac{1}{5} - \frac{1}{3}\right) = -\frac{32\pi}{15}$$

$$\bar{z} = \frac{z - mom}{V} = -\frac{32\pi}{15}\frac{3}{8\pi} = -\frac{4}{5}$$

12. (10 points) Compute $\iint_R y^2 \, dx\,dy$ over the diamond shaped region R bounded by

$$y = \frac{1}{x}, \qquad y = \frac{6}{x}, \qquad y = x, \qquad y = 2x$$



FULL CREDIT for integrating in the curvilinear coordinates (u,v) where $u^2 = xy$ and $v^2 = \frac{y}{x}$. (Solve for x and y.)

HALF CREDIT for integrating in rectangular coordinates.

$$\left\{ \begin{array}{l} u^2 = xy \\ v^2 = \frac{y}{x} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} u^2 v^2 = y^2 \\ \frac{u^2}{v^2} = x^2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x = \frac{u}{v} \\ y = uv \end{array} \right\}$$

$$J = \left| \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \right| = \left| \left| \begin{array}{cc} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{array} \right| \right| = \left| \frac{u}{v} - -\frac{u}{v} \right| = \frac{2u}{v}$$

$$xy = 1 \Rightarrow u^2 = 1 \Rightarrow u = 1 \qquad xy = 6 \Rightarrow u^2 = 6 \Rightarrow u = \sqrt{6} \qquad \text{So: } 1 \le u \le \sqrt{6}$$

$$\frac{y}{x} = 1 \Rightarrow v^2 = 1 \Rightarrow v = 1 \qquad \frac{y}{x} = 2 \Rightarrow v^2 = 2 \Rightarrow v = \sqrt{2} \qquad \text{So: } 1 \le v \le \sqrt{2}$$

$$\iint_R y^2 \, dx\,dy = \int_1^{\sqrt{2}} \int_1^{\sqrt{6}} u^2 v^2 \, \frac{2u}{v} \, du \, dv = 2 \int_1^{\sqrt{2}} \int_1^{\sqrt{6}} u^3 v \, du \, dv$$

$$= 2\left[\frac{u^4}{4} \right]_{u=1}^{\sqrt{6}} \left[\frac{v^2}{2} \right]_{v=1}^{\sqrt{2}} = 2\left[\frac{36}{4} - \frac{1}{4} \right]\left[\frac{2}{2} - \frac{1}{2} \right] = \frac{35}{4}$$