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MATH 253 EXAM 3 Fall 1998

Sections 501-503 Solutions P. Yasskin

Multiple Choice: (7 points each)

1. If $F = \left(yz\cos x,\ y\sin x,\ z\sin x\right)$ then $\vec{\nabla}\bullet\vec{F} =$

a. $-yz\sin x$

b. $(2-yz)\sin x$ correctchoice

c. $\left(-yz\sin x,\ -\sin x,\ \sin x\right)$

d. $\left(0,\ (z-y)\cos x,\ (y-z)\cos x\right)$

e. $\left(-yz\sin x,\ \sin x,\ \sin x\right)$

$$\vec{\nabla}\bullet\vec{F} = \frac{\partial}{\partial x}(yz\cos x) + \frac{\partial}{\partial y}(y\sin x) + \frac{\partial}{\partial z}(z\sin x) = -yz\sin x + \sin x + \sin x = (2-yz)\sin x$$

2. If $F = \left(yz\cos x,\ y\sin x,\ z\sin x\right)$ then $\vec{\nabla}\times\vec{F} =$

a. $\left(0,\ (y-z)\cos x,\ (y-z)\cos x\right)$ correctchoice

b. $2(y-z)\sin x$

c. $\left(-yz\sin x,\ -\sin x,\ \sin x\right)$

d. $\left(0,\ (z-y)\cos x,\ (y-z)\cos x\right)$

e. $\vec{0}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz\cos x & y\sin x & z\sin x \end{vmatrix} = \hat{\imath}(0-0) - \hat{\jmath}(z\cos x - y\cos x) + \hat{k}(y\cos x - z\cos x)$$

$$= \left(0,\ (y-z)\cos x,\ (y-z)\cos x\right)$$

3. Find a scalar potential for $F = \left(yz\cos x,\ y\sin x,\ z\sin x\right)$.

a. $-yz\sin x$

b. $yz\sin x$

c. $\left(yz\sin x,\ \frac{y^2}{2}\sin x,\ \frac{z^2}{2}\sin x\right)$

d. $yz\sin x + \frac{y^2}{2}\sin x + \frac{z^2}{2}\sin x$

e. Does Not Exist correctchoice

Since $\vec{\nabla}\times\vec{F} \neq \vec{0}$, there is no potential.

4. Compute the line integral $\int_{A \atop H}^{B} \vec{F} \bullet d\vec{s}$ of the vector field $\vec{F} = (y, -x, z)$ along the helix H parametrized by $\vec{r}(t) = (3\cos\theta, 3\sin\theta, \theta)$ between $A = (3,0,0)$ and $B = (-3,0,3\pi)$.

- **a.** $2\pi^2 - 18\pi$
- **b.** 2π
- **c.** 3π
- **d.** $\dfrac{9\pi^2}{2} - 27\pi$ correctchoice
- **e.** $\dfrac{9\pi^2}{2}$

$\vec{v} = (-3\sin\theta, 3\cos\theta, 1)$ $\quad\quad$ $\vec{F} = (y, -x, z) = (3\sin\theta, -3\cos\theta, \theta)$

$$\int_{A \atop H}^{B} \vec{F} \bullet d\vec{s} = \int_0^{3\pi} \vec{F} \bullet \vec{v}\, d\theta = \int_0^{3\pi} \left(-9\sin^2\theta - 9\cos^2\theta + \theta\right) d\theta = \int_0^{3\pi} (-9 + \theta)\, d\theta$$

$$= \left[-9\theta + \frac{\theta^2}{2} \right]_0^{3\pi} = -27\pi + \frac{9\pi^2}{2}$$

5. Find the total mass of the helix H parametrized by $\vec{r}(t) = (3\cos\theta, 3\sin\theta, \theta)$ between $A = (3,0,0)$ and $B = (-3,0,3\pi)$ if the linear mass density is $\rho = 3 + 2z$.

- **a.** $9(\pi + \pi^2)$
- **b.** $27(\pi + \pi^2)$
- **c.** $9\sqrt{10}\,(\pi + \pi^2)$ correctchoice
- **d.** $\sqrt{10}\,(6\pi + 4\pi^2)$
- **e.** $6\pi + 4\pi^2$

$\vec{r}$ and hence $\vec{v}$ are the same as in #4. So: $\quad |\vec{v}| = \sqrt{9\sin^2\theta + 9\cos^2\theta + 1} = \sqrt{10}$

$\rho = 3 + 2z = 3 + 2\theta \quad\quad M = \int_0^{3\pi} (3 + 2\theta)\sqrt{10}\; d\theta = \sqrt{10}\left[3\theta + \theta^2\right]_0^{3\pi} = \sqrt{10}\,(9\pi + 9\pi^2)$

6. Compute $\oint (4x - 3y)\,dx + (3x - 2y)\,dy$ counterclockwise around the edge of the triangle with vertices $(0,0)$, $(0,3)$ and $(2,0)$.
(HINT: Use Green's Theorem.)

- **a.** 3
- **b.** 6
- **c.** 9
- **d.** 12
- **e.** 18 correctchoice

$$\oint P\,dx + Q\,dy = \iint_{\text{triangle}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_{\text{triangle}} (3 - -3)\; dx\,dy = 6\left(\text{area of triangle}\right)$$

$$= 6 \bullet \frac{1}{2} \bullet 2 \bullet 3 = 18$$

7. Compute $\iint_P \vec{\nabla} \times \vec{F} \bullet d\vec{S}$ for the vector field $\vec{F} = (-y, x, z)$ over the paraboloid $z = x^2 + y^2$ for $z \le 9$ with normal pointing in and up.
(HINT: Use Stokes' Theorem.)

- **a.** 2π
- **b.** 3π
- **c.** 4π
- **d.** 9π
- **e.** 18π correctchoice

By Stokes', $\iint_P \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \oint_{\partial P} \vec{F} \bullet d\vec{s}$ So we parametrize the boundary circle:

$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 9)$ $\vec{v} = (-3\sin\theta, 3\cos\theta, 0)$ $\vec{F} = (-y, x, z) = (-3\sin\theta, 3\cos\theta, 9)$

$$\iint_P \vec{\nabla} \times \vec{F} \bullet d\vec{S} = \int_0^{2\pi} \vec{F} \bullet \vec{v}\ d\theta = \int_0^{2\pi} 9\ d\theta = 18\pi$$

8. (25 points) Green's Theorem states that if R is a nice region in the plane and ∂R is its boundary curve traversed counterclockwise then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \oint_{\partial R} P\,dx + Q\,dy$$

Verify Green's Theorem if $P = -x^2 y$ and $Q = xy^2$ and R is the region inside the circle $x^2 + y^2 = 4$.

a. (5 pts) Compute $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. (HINT: Use rectangular coordinates.)

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(-x^2 y) = y^2 + x^2$$

b. (10 pts) Compute $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$.

(HINT: Switch to polar coordinates and don't forget the Jacobian.)

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_R (x^2 + y^2)\ dx\,dy = \int_0^{2\pi} \int_0^2 (r^2)\ r\ dr\,d\theta = 2\pi \left[\frac{r^4}{4} \right]_0^2 = 8\pi$$

c. (10 pts) Compute $\oint_{\partial R} P\,dx + Q\,dy$. (HINT: Parametrize the boundary circle.)

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta)$ $\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta)$

$\vec{F} = (-x^2 y, xy^2) = \left(-8\cos^2\theta\sin\theta, 8\cos\theta\sin^2\theta\right)$

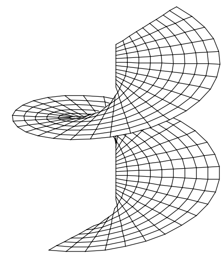
$\vec{F} \bullet \vec{v} = 16\cos^2\theta\sin^2\theta + 16\cos^2\theta\sin^2\theta = 32\cos^2\theta\sin^2\theta = 8(2\sin\theta\cos\theta)^2$

$= 8\sin^2(2\theta) = 8\frac{1 - \cos(4\theta)}{2} = 4 - 4\cos(4\theta)$

$$\oint_{\partial R} P\,dx + Q\,dy = \int_0^{2\pi} 4 - 4\cos(4\theta)\ d\theta = \left[4\theta - \sin(4\theta) \right]_0^{2\pi} = 8\pi$$

Note: The answers to (b) and (c) are the same.

9. (15 points) The spiral ramp at the right may be parametrized as



$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$$

for $0 \le r \le 2$ and $0 \le \theta \le 3\pi$.

Compute $\iint \sqrt{x^2+y^2}\ dS$ over this spiral ramp.

$$\begin{array}{rlll} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{R}_r = & (\cos\theta, & \sin\theta, & 0) \\ \vec{R}_\theta = & (-r\sin\theta, & r\cos\theta, & 1) \\ \vec{N} = & \hat{\imath}(\sin\theta) - & \hat{\jmath}(\cos\theta) & + \hat{k}(r) \end{array}$$

$$\left|\vec{N}\right| = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{1+r^2}$$

$$\iint \sqrt{x^2+y^2}\ dS = \int_0^{3\pi}\int_0^2 r\sqrt{1+r^2}\ dr\,d\theta$$

$$= 3\pi\left[\frac{1}{2}\frac{2}{3}(1+r^2)^{3/2}\right]_0^2 = \pi(5^{3/2}-1)$$

10. (25 points) Gauss' Theorem states that if V is a solid region and ∂V is its boundary surface with outward normal then

$$\iiint\limits_V \vec{\nabla}\bullet\vec{F}\ dV = \iint\limits_{\partial V} \vec{F}\bullet d\vec{S}$$

Verify Gauss' Theorem if $\vec{F} = (xz^2, yz^2, z^3)$ and V is the solid region above the paraboloid $z = x^2+y^2$ below the plane $z = 4$. Notice that ∂V consists of two parts: a piece of the paraboloid P and a disk D.

a. (7 pts) Compute $\iiint\limits_V \vec{\nabla}\bullet\vec{F}\ dV$. (HINTS: Compute $\vec{\nabla}\bullet\vec{F}$ in rectangular coordinates. Integrate in cylindrical coordinates.)

$$\vec{\nabla}\bullet\vec{F} = z^2 + z^2 + 3z^2 = 5z^2$$

$$\iiint\limits_V \vec{\nabla}\bullet\vec{F}\ dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 5z^2 r\ dz\,dr\,d\theta = 2\pi\int_0^2\left[5\frac{z^3}{3}r\right]_{z=r^2}^4 dr = 2\pi\frac{5}{3}\int_0^2[64r - r^7]dr$$

$$= \frac{10\pi}{3}\left[32r^2 - \frac{r^8}{8}\right]_0^2 = \frac{10\pi}{3}[128-32] = \frac{10\pi}{3}[96] = 320\pi$$

b. (8 pts) Compute $\iint_P \vec{F} \bullet d\vec{S}$. (HINT: Here is the parametrization of the paraboloid.)

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$$

$$\begin{array}{lllll} & \hat{\imath} & \hat{\jmath} & \hat{k} & \\ \vec{R}_r = & (\cos\theta, & \sin\theta, & 2r) & \\ \vec{R}_\theta = & (-r\sin\theta, & r\cos\theta, & 0\ ) & \\ \vec{N} = & \hat{\imath}(-2r^2\cos\theta) - & \hat{\jmath}(--2r^2\sin\theta) & +\ \hat{k}(r) & = (-2r^2\cos\theta, -2r^2\sin\theta, r) \end{array}$$

$$\vec{F}\left(\vec{R}(r,\theta)\right) = (xz^2, yz^2, z^3) = (r^5\cos\theta, r^5\sin\theta, r^6)$$

$$\iint_P \vec{F} \bullet d\vec{S} = \int_0^{2\pi}\int_0^2 \vec{F} \bullet \vec{N}\ dr\,d\theta = \int_0^{2\pi}\int_0^2 \left(-2r^7\cos^2\theta - 2r^7\sin^2\theta + r^7\right)\ dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 -r^7\ dr\,d\theta = 2\pi\left[-\frac{r^8}{8}\right]_0^2 = -2\pi(32) = -64\pi$$

c. (7 pts) Compute $\iint_D \vec{F} \bullet d\vec{S}$. (HINT: You parametrize the disk.)

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 4)$$

$$\begin{array}{llll} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{R}_r = & (\cos\theta, & \sin\theta, & 0\ ) \\ \vec{R}_\theta = & (-r\sin\theta, & r\cos\theta, & 0\ ) \\ \vec{N} = & (0, & 0, & r) \end{array}$$

$$\vec{F}\left(\vec{R}(r,\theta)\right) = (xz^2, yz^2, z^3) = (16r\cos\theta, 16r\sin\theta, 64)$$

$$\iint_D \vec{F} \bullet d\vec{S} = \int_0^{2\pi}\int_0^2 \vec{F} \bullet \vec{N}\ dr\,d\theta = \int_0^{2\pi}\int_0^2 64r\ dr\,d\theta = 2\pi[32r^2]_0^2 = 265\pi$$

d. (3 pts) Combine $\iint_P \vec{F} \bullet d\vec{S}$ and $\iint_D \vec{F} \bullet d\vec{S}$. to obtain $\iint_{\partial V} \vec{F} \bullet d\vec{S}$.

Be sure to discuss the orientations of the surfaces and to give a formula before you plug in numbers.

The normal to ∂V must point outward. The normal to P points in and up but should point down and out. So an extra minus is needed. The normal to D points up which is correct. So:

$$\iint_{\partial V} \vec{F} \bullet d\vec{S} = -\iint_P \vec{F} \bullet d\vec{S} + \iint_D \vec{F} \bullet d\vec{S} = --64\pi + 256\pi = 320\pi$$

Note: The answers to (a) and (d) are the same.