Name______________ ID__________ Section_____	1-4	/40	7	/15
MATH 253 Honors EXAM 3 Fall 2002	5	/15	8	/15
Sections 201-202 Solutions P. Yasskin	6	/15	E.C.	/10

Multiple Choice: (10 points each) Work Out: (15 points each) Extra Credit: (10 points)

1. If $\vec{\nabla}\times\vec{F} = (2x^2y - x^2, y^2 - 2xy^2, 2xz - 2yz)$, then $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{F}\right) =$

a. $(4xy - 2x, 4xy - 2y, 2x - 2y)$
b. $(4xy - 2x, 2y - 4xy, 2x - 2y)$
c. $8xy - 4y$
d. 0 correctchoice
e. $2x^2 - 2y^2$

For any vector field $\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} = 0$

2. If $\vec{\nabla}\times\vec{F} = (2x^2y - x^2, y^2 - 2xy^2, 2xz - 2yz)$, then $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right) =$

a. $(4xy - 2x, 4xy - 2y, 2x - 2y)$
b. $(4xy - 2x, 2y - 4xy, 2x - 2y)$
c. $(-2z, -2z, -2x^2 - 2y^2)$ correctchoice
d. $(-2z, 2z, -2x^2 - 2y^2)$
e. $-2x^2 - 2y^2$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2x^2y - x^2 & y^2 - 2xy^2 & 2xz - 2yz \end{vmatrix} = \hat{\imath}(-2z) - \hat{\jmath}(2z) + \hat{k}(-2y^2 - 2x^2)$$

3. If $\vec{G} = (2x^2y - x^2, y^2 - 2xy^2, 2xz - 2yz)$, then $\vec{G} = \vec{\nabla}g$ where $g(0,1,1) - g(0,1,0) =$

a. -2
b. -1
c. 0
d. 1
e. The scalar potential g does not exist. correctchoice

Since $\vec{\nabla}\times\vec{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2x^2y - x^2 & y^2 - 2xy^2 & 2xz - 2yz \end{vmatrix} = \hat{\imath}(-2z) - \hat{\jmath}(2z) + \hat{k}(-2y^2 - 2x^2) \neq (0,0,0)$,
the scalar potential g does not exist.

4. Compute $\oint (y+z)\,dx + (x+z)\,dy + (x+y)\,dz$ clockwise around the circle $x^2 + y^2 = 9$ with $z = 5$.

HINT: Use a theorem.

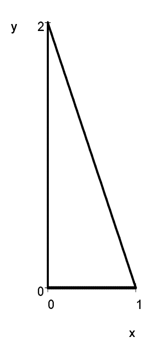
a. -18π
b. -9π
c. 0 correctchoice
d. 9π
e. 18π

Let $\vec{F} = (y+z, x+z, x+y)$. Then $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y+z & x+z & x+y \end{vmatrix} = (0,0,0)$.

By Stokes' Theorem, $\oint \vec{F} \cdot d\vec{s} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = 0$

where the surface integral is over the disk bounded by the circle.

5. Compute $\oint_{\partial T} (xy)\,dx + (xy)\,dy$ counterclockwise around the boundary of the triangle with vertices $(0,0)$, $(1,0)$ and $(0,2)$.



By Green's theorem,

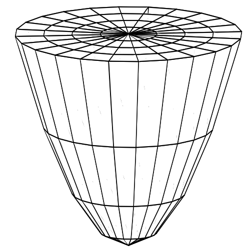
$$\oint_{\partial T} (xy)\,dx + (xy)\,dy = \iint_T \left(\frac{\partial(xy)}{\partial x} - \frac{\partial(xy)}{\partial y} \right) dx\,dy$$

$$= \int_0^1 \int_0^{2-2x} (y-x)\,dy\,dx = \int_0^1 \left[\frac{y}{2}^2 - xy \right]_0^{2-2x} dx$$

$$= \int_0^1 \left[\frac{(2-2x)}{2}^2 - x(2-2x) \right] dx = \int_0^1 (2 - 4x + 2x^2 - 2x + 2x^2)\,dx$$

$$= \int_0^1 (2 - 6x + 4x^2)\,dx = \left[2x - 3x^2 + 4\frac{x^3}{3} \right]_0^1 = \left[2 - 3 + \frac{4}{3} \right] = \frac{1}{3}$$

6. Compute $\iint\limits_{\partial P} \vec{E} \cdot d\vec{S}$ for $\vec{E} = (xz, yz, z^2)$ over the **complete** surface of the solid paraboloid P given by $x^2 + y^2 \le z \le 4$ with outward normal.



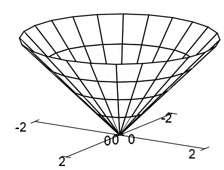
By Gauss' Theorem, $\iint\limits_{\partial P} \vec{E} \cdot d\vec{S} = \iiint\limits_{P} \vec{\nabla} \cdot \vec{E}\, dV = \iiint\limits_{P} (z + z + 2z)\, dV = \iiint\limits_{P} (4z)\, dV$

In cylindrical coordinates, the paraboloid is $r^2 \le z \le 4$. So

$$\iint\limits_{\partial P} \vec{E} \cdot d\vec{S} = \int_0^2 \int_0^{2\pi} \int_{r^2}^4 (4z)\, r\, dz\, d\theta\, dr = 2\pi \int_0^2 [2z^2]_{z=r^2}^4\, r\, dr = 2\pi \int_0^2 (32 - 2r^4)\, r\, dr$$

$$= 2\pi \left[32\frac{r^2}{2} - 2\frac{r^6}{6} \right]_0^2 = 2\pi \left(64 - \frac{64}{3} \right) = 128\pi \left(1 - \frac{1}{3} \right) = \frac{256\pi}{3}$$

7. The cone $z = \sqrt{x^2 + y^2}$ for $z \le 2$ is shown at the right. Find the mass and center of mass of the cone if its surface density is given by $\delta = x^2 + y^2$.



$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r) \qquad \vec{e}_r = (\cos\theta, \sin\theta, 1) \qquad \vec{e}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = (-r\cos\theta, -r\sin\theta, r) \qquad \left|\vec{N}\right| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2}\, r$$

$$\delta = x^2 + y^2 = r^2$$

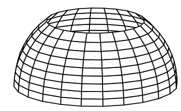
$$M = \iint\limits_C \delta\, dS = \iint\limits_C \delta\left(\vec{R}(r,\theta)\right) \left|\vec{N}\right| dr\, d\theta = \int_0^{2\pi} \int_0^2 r^2 \sqrt{2}\, r\, dr\, d\theta = 2\pi\sqrt{2}\, \frac{r^4}{4} \Big|_0^2 = 8\pi\sqrt{2}$$

By symmetry, $\bar{x} = \bar{y} = 0$.

$$z\text{-mom} = \iint\limits_C z\,\delta\, dS = \iint\limits_C z\,\delta\left(\vec{R}(r,\theta)\right) \left|\vec{N}\right| dr\, d\theta = \int_0^{2\pi} \int_0^2 r^3 \sqrt{2}\, r\, dr\, d\theta = 2\pi\sqrt{2}\, \frac{r^5}{5} \Big|_0^2 = \frac{64\pi\sqrt{2}}{5}$$

$$\bar{z} = \frac{z\text{-mom}}{M} = \frac{64\pi\sqrt{2}}{5 \cdot 8\pi\sqrt{2}} = \frac{8}{5}$$

8. Compute $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ for $\vec{F} = (x^2y, -x^3, z^2)$



over the piece of the sphere $x^2 + y^2 + z^2 = 25$

for $0 \le z \le 4$ with normal pointing away from the z-axis.

Hint: Parametrize the upper and lower edges.

By Stokes' Theorem $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s} = \oint_{\text{upper}} \vec{F} \cdot d\vec{s} + \oint_{\text{lower}} \vec{F} \cdot d\vec{s}$

By the right hand rule, since the normal points outward, the **upper** circle must be traversed **clockwise** while the **lower** circle must be traversed **counterclockwise** as seen from the positive z-axis. We compute each line integral:

Upper Circle: $z = 4 \qquad x^2 + y^2 = 25 - z^2 = 25 - 16 = 9$

$\vec{r}(t) = (3\cos t, 3\sin t, 4) \qquad \vec{v} = (-3\sin t, 3\cos t, 0)$

This is clockwise, so we reverse the velocity: $\vec{v} = (3\sin t, -3\cos t, 0)$

$\vec{F} = (x^2y, -x^3, z^2) = (27\cos^2 t \sin t, -27\cos^3 t, 16)$

$$\oint_{\text{upper}} \vec{F} \cdot d\vec{s} = \oint_{\text{upper}} \vec{F} \cdot \vec{v}\, dt = \int_0^{2\pi} (81\cos^2 t \sin^2 t + 81\cos^4 t)\, dt = 81\int_0^{2\pi} \cos^2 t(\sin^2 t + \cos^2 t)\, dt$$

$$= 81\int_0^{2\pi} \frac{1 + \cos 2t}{2}\, dt = \frac{81}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = 81\pi$$

Lower Circle: $z = 0 \qquad x^2 + y^2 = 25 - z^2 = 25$

$\vec{r}(t) = (5\cos t, 5\sin t, 0) \qquad \vec{v} = (-5\sin t, 5\cos t, 0)$

This is counterclockwise, so we do not need to reverse the velocity.

$\vec{F} = (x^2y, -x^3, z^2) = (125\cos^2 t \sin t, -125\cos^3 t, 0)$

$$\oint_{\text{lower}} \vec{F} \cdot d\vec{s} = \oint_{\text{lower}} \vec{F} \cdot \vec{v}\, dt = \int_0^{2\pi} (-625\cos^2 t \sin^2 t - 625\cos^4 t)\, dt = -625\int_0^{2\pi} \cos^2 t(\sin^2 t + \cos^2 t)\, dt$$

$$= -625\int_0^{2\pi} \frac{1 + \cos 2t}{2}\, dt = \frac{-625}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = -625\pi$$

Total Boundary:

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = 81\pi - 625\pi = -544\pi$$

Extra Credit Redo #8 but compute $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ directly as a surface integral.

Use spherical coordinates:

$$\vec{R}(\varphi,\theta) = (5\sin\varphi\cos\theta, 5\sin\varphi\sin\theta, 5\cos\varphi) \qquad \arccos\frac{4}{5} \le \varphi \le \frac{\pi}{2} \qquad 0 \le \theta \le 2\pi$$

$$\vec{e}_\varphi = (5\cos\varphi\cos\theta, 5\cos\varphi\sin\theta, -5\sin\varphi) \qquad \vec{e}_\theta = (-5\sin\varphi\sin\theta, 5\sin\varphi\cos\theta, 0)$$

$$\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = (25\sin^2\varphi\cos\theta, 25\sin^2\varphi\sin\theta, 25\sin\varphi\cos\varphi)$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2y & -x^3 & z^2 \end{vmatrix} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(-3x^2 - x^2) = (0,0,-4x^2) = (0,0,-100\sin^2\varphi\cos^2\theta)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = -2500\sin^3\varphi\cos\varphi\cos^2\theta$$

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_S \vec{\nabla} \times \vec{F} \cdot \vec{N}\, d\varphi\, d\theta = \int_0^{2\pi} \int_{\arccos 4/5}^{\pi/2} -2500\sin^3\varphi\cos\varphi\cos^2\theta\, d\varphi\, d\theta$$

$$= -2500 \int_0^{2\pi} \cos^2\theta\, d\theta \int_{\arccos 4/5}^{\pi/2} \sin^3\varphi\cos\varphi\, d\varphi = -2500\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi} \left[\frac{\sin^4\varphi}{4}\right]_{\arccos 4/5}^{\pi/2}$$

$$= -\frac{2500}{4}[\pi][1 - \sin^4 \arccos 4/5] = -\frac{2500\pi}{4}\left[1 - \left(\frac{3}{5}\right)^4\right] = -\frac{2500\pi}{4}\left(\frac{625 - 81}{625}\right) = -544\pi$$