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MATH 221/253H Exam 1 Fall 2012

Sections 502,201,202 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the area of the triangle whose vertices are
$P = (2,4,-3)$, $Q = (3,4,-2)$ and $R = (0,6,-3)$.

a. 12

b. 6

c. $2\sqrt{3}$

d. $\sqrt{3}$ Correct Choice

e. 1

SOLUTION: $\overrightarrow{PQ} = Q - P = \langle 1,0,1 \rangle$ $\overrightarrow{PR} = R - P = \langle -2,2,0 \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 1 \\ -2 & 2 & 0 \end{vmatrix} = \hat{\imath}(0-2) - \hat{\jmath}(0--2) + \hat{k}(2-0) = \langle -2,-2,2 \rangle$$

$$A = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{4+4+4} = \sqrt{3}$$

2. Which of the following is the hyperplane in $\mathbb{R}^4$ which contains the point $P = (x,y,z,w) = (8,4,2,1)$ and is tangent to the vectors $\vec{p} = \langle 2,1,0,0 \rangle$ $\vec{q} = \langle 0,2,1,0 \rangle$ and $\vec{r} = \langle 0,0,2,1 \rangle$?
HINT: What is the vector perpendicular to the hyperplane?

a. $8x + 4y + 2z + w = 85$

b. $8x - 4y + 2z - w = 51$

c. $x + 2y + 4z + 8w = 32$

d. $x - 2y + 4z - 8w = 0$ Correct Choice

e. $x - y + z - w = 5$

SOLUTION:

$$\vec{N} = \perp(\vec{p},\vec{q},\vec{r}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} & \hat{l} \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{vmatrix} = \hat{\imath}\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} - \hat{\jmath}\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \hat{l}\begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{vmatrix}$$

$$\vec{N} = \hat{\imath}(1-0) - \hat{\jmath}(2-0) + \hat{k}(4-0) - \hat{l}(8-0) = \langle 1,2,4,8 \rangle$$

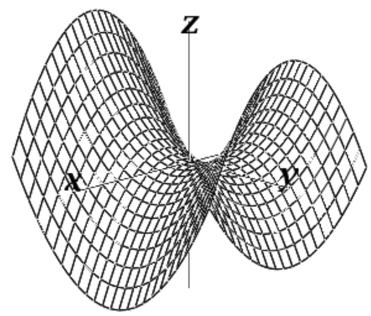
$$\vec{N} \bullet X = \vec{N} \bullet P \qquad x - 2y + 4z - 8w = 1 \bullet 8 - 2 \bullet 4 + 4 \bullet 2 - 8 \bullet 1 = 0$$

3. The quadratic surface $x^2 + y^2 - z^2 + 4x + 4y - 6z = 0$ is

a. a cone.
b. an elliptic hyperboloid
c. a hyperbolic paraboloid
d. a hyperboloid of 1 sheet
e. a hyperboloid of 2 sheets Correct Choice

SOLUTION: We complete the squares to get
$(x+2)^2 + (y+2)^2 - (z+3)^2 = 4 + 4 - 9 = -1$
Since there are squares on each variable with different signs and non-zero on the right, it is a hyperboloid. Rearranging, we have
$(z+3)^2 = 1 + (x+2)^2 + (y+2)^2$
So $(z+3)^2$ can never be zero and it must be a hyperboloid of 2 sheets.

4. The plot at the right is the graph of which equation?



a. $z = x^2 - y^2$
b. $z = -x^2 + y^2$ Correct Choice
c. $z^2 = x^2 - y^2$
d. $z^2 = -x^2 + y^2$
e. $z^2 - x^2 - y^2 = 1$

SOLUTION: The saddle surface is a hyperbolic paraboloid: (a) or (b).

Since it goes down in the x-direction and up in the y-direction, it is (a).

5. An airplane is travelling due South with constant speed and constant altitude as it flies over College Station. Since its path is part of a circle around the earth, its acceleration points directly toward the center of the earth. In which direction does it binormal $\hat{B}$ point?

a. Up

b. North

c. South

d. East Correct Choice

e. West

SOLUTION: $\vec{v}$ is South. $\vec{a}$ is Down.
So $\hat{B} = \dfrac{\hat{v} \times \hat{a}}{|\hat{v} \times \hat{a}|}$ points East by the right hand rule.

6. For the curve $\vec{r}(t) = (4\cos t,\quad 3t,\quad 4\sin t)$ which of the following is FALSE?

a. $\vec{v} = \langle -4\sin t,\quad 3,\quad 4\cos t\rangle$

b. $\vec{a} = \langle -4\cos t,\quad 0,\quad -4\sin t\rangle$

c. $|\vec{v}| = 25$ Correct Choice

d. Arc length between $t = 0$ and $t = 2\pi$ is 10π

e. $a_T = 0$

SOLUTION: $\vec{v}$ and $\vec{a}$ are correct by differentiation.

$$|\vec{v}| = \sqrt{9 + 16\sin^2 t + 16\cos^2 t} = \boxed{5} \qquad a_T = \frac{d|\vec{v}|}{dt} = 0$$

$$L = \int ds = \int |\vec{v}|\, dt = \int_0^{2\pi} 5\, dt = [5t]_0^{2\pi} = 10\pi$$

7. A wire in the shape of the curve $\vec{r}(t) = (4\cos t,\quad 3t,\quad 4\sin t)$ has linear mass density $\rho = y + z$. Find its total mass between $t = 0$ and $t = 2\pi$.

a. $30\pi^2$ Correct Choice

b. $6\pi^2$

c. 30π

d. 12π

e. 6π

SOLUTION: $M = \int \rho\, ds = \int (y + z)|\vec{v}|\, dt = \int_0^{2\pi} (3t + 4\sin t)5\, dt = \left[\frac{15t^2}{2} - 20\cos t\right]_0^{2\pi} = 30\pi^2$

8. Find the work done to move an object along the curve $\vec{r}(t) = (4\cos t,\quad 3t,\quad 4\sin t)$ between $t = 0$ and $t = 2\pi$ by the force $\vec{F} = \langle z, 0, -x\rangle$?

a. 32π

b. 25π

c. $-25\pi^2$

d. -25π

e. -32π Correct Choice

SOLUTION: $\vec{F}(\vec{r}(t)) = \langle 4\sin t,\quad 0,\quad -4\cos t\rangle \qquad \vec{v} = \langle -4\sin t,\quad 3,\quad 4\cos t\rangle$

$$W = \int \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{v}\, dt = \int_0^{2\pi} (-16\sin^2 t - 16\cos^2 t)\, dt = \int_0^{2\pi} -16\, dt = -32\pi$$

9. Find the plane tangent to the graph of $z = xe^y$ at the point $(2,0)$. Its z-intercept is

a. $-e$

b. -2

c. 0 Correct Choice

d. 2

e. e

SOLUTION:

$f = xe^y$ $\quad f(2,0) = 2$ $\quad z = f(2,0) + f_x(2,0)(x-2) + f_y(2,0)(y-0)$

$f_x = e^y$ $\quad f_x(2,0) = 1$ $\quad = 2 + 1(x-2) + 2y$

$f_y = xe^y$ $\quad f_y(2,0) = 2$ $\quad$ When $x = y = 0$, we have $z = 2 + (-2) = 0$.

10. Find the plane tangent to the graph of $xz^3 + zy^2 + yx^4 = 42$ at the point $(1,2,0)$. Its z-intercept is

a. $\frac{2}{5}$

b. $\frac{4}{5}$

c. $\frac{5}{4}$

d. $\frac{5}{2}$ Correct Choice

e. 10

SOLUTION: $F(x,y,z) = xz^3 + zy^2 + yx^4$ $\quad \vec{\nabla}F = \langle z^3 + 4yx^3, 2zy + x^4, 3xz^2 + y^2 \rangle$

$\vec{N} = \left|\vec{\nabla}F\right|_{(1,2,0)} = \langle 8,1,4 \rangle$ $\quad \vec{N} \cdot X = \vec{N} \cdot P$ $\quad 8x + y + 4z = 8 \cdot 1 + 2 + 4 \cdot 0 = 10$

When $x = y = 0$, we have $z = \frac{5}{2}$.

11. Hans Duo is currently at $(x,y,z) = (3,2,1)$ and flying the Milenium Eagle through a deadly polaron field whose density is $\rho = x^2z + yz^2$. In what unit vector direction should he travel to **reduce** the density as fast as possible?

a. $\langle -6,-1,-13\rangle$

b. $\frac{1}{\sqrt{206}}\langle -6,-1,-13\rangle$ Correct Choice

c. $\langle 6,1,13\rangle$

d. $\frac{1}{\sqrt{206}}\langle -6,1,-13\rangle$

e. $\frac{1}{\sqrt{206}}\langle 6,-1,13\rangle$

SOLUTION: $\vec{\nabla}\rho = \langle 2xz, z^2, x^2 + 2yz\rangle$ $\quad \vec{v} = -\vec{\nabla}\rho\Big|_{(3,2,1)} = \langle -6,-1,-13\rangle$

$|\vec{v}| = \sqrt{36+1+169} = \sqrt{206}$ $\quad \hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{206}}\langle -6,-1,-13\rangle$

12. The point $(x,y) = (9,3)$ is a critical point of the function $f(x,y) = x^2 - 2xy^2 + 4y^3$. Use the Second Derivative Test to classify this critical point.

a. local minimum

b. local maximum

c. saddle point Correct Choice

d. TEST FAILS

SOLUTION:

$f_x = 2x - 2y^2 \quad \Rightarrow \quad f_x(9,3) = 18 - 18 = 0$ Checked

$f_y = -4xy + 12y^2 \quad \Rightarrow \quad f_y(9,3) = -4 \cdot 27 + 12 \cdot 9 = 0$ Checked

$f_{xx} = 2 \quad \Rightarrow \quad f_{xx}(9,3) = 2$

$f_{yy} = -4x + 24y \quad \Rightarrow \quad f_{yy}(9,3) = -36 + 72 = 36$

$f_{xy} = -4y \quad \Rightarrow \quad f_{xy}(9,3) = -12$

$D = f_{xx}f_{yy} - {f_{xy}}^2 = 2 \cdot 36 - 12^2 = -72$

Since $D < 0$ it is a saddle point.

Work Out: (10 points each. Part credit possible. Show all work.)

13. Find the scalar and vector projections of the vector $\vec{a} = \langle 2, -1, 2 \rangle$ along the vector $\vec{b} = \langle 1, 2, -2 \rangle$.

SOLUTION: $\vec{a} \cdot \vec{b} = 2 - 2 - 4 = -4$ $\vec{b} \cdot \vec{b} = 4 + 1 + 4 = 9$ $|\vec{b}| = 3$

$$\text{comp}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{-4}{3}$$

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}\vec{b} = \frac{-4}{9}\langle 1, 2, -2 \rangle = \left\langle \frac{-4}{9}, \frac{-8}{9}, \frac{8}{9} \right\rangle$$

14. The pressure, P, volume, V, and temperature, T, of an ideal gas are related by

$P = \frac{kT}{V}$ for some constant k. For a certain sample $k = 10 \frac{\text{cm}^3\text{-atm}}{°\text{K}}$.

At a certain instant, the volume and temperature are $V = 2000\,\text{cm}^3$, and $T = 300\,°\text{K}$,

and are increasing at $\frac{dV}{dt} = 40 \frac{\text{cm}^3}{\text{sec}}$, and $\frac{dT}{dt} = 5 \frac{°\text{K}}{\text{sec}}$.

At that instant, what is the pressure, is it increasing or decreasing and at what rate?

SOLUTION: $P = \frac{kT}{V} = \frac{10 \cdot 300}{2000} \frac{\text{cm}^3\text{-atm}}{°\text{K}} \frac{°\text{K}}{\text{cm}^3} = 1.5\,\text{atm}$

$$\frac{dP}{dt} = \frac{\partial P}{\partial V}\frac{dV}{dt} + \frac{\partial P}{\partial T}\frac{dT}{dt} = \frac{-kT}{V^2}\frac{dV}{dt} + \frac{k}{V}\frac{dT}{dt} = \frac{-10 \cdot 300}{2000^2} \cdot 40 + \frac{10}{2000}5 = -\frac{5}{1000} = -0.005 \frac{\text{atm}}{\text{sec}}$$

Since $\frac{dP}{dt}$ is negative, the pressure is decreasing.

15. If two resistors, with resistances R_1 and R_2, are arranged in parallel, the total resistance R is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \qquad \text{or} \qquad R = \frac{R_1 R_2}{R_1 + R_2}$$

If $R_1 = 4\Omega$ and $R_2 = 6\Omega$ and the uncertainty in the measurement of R_1 is $\Delta R_1 = 0.03\Omega$ and for R_2 is $\Delta R_2 = 0.02\Omega$. Find R and use differentials to estimate the uncertainty in the measurment of R.

SOLUTION: $R = \frac{4 \cdot 6}{4+6} = 2.4\Omega$.

$$\Delta R = \frac{\partial R}{\partial R_1}\Delta R_1 + \frac{\partial R}{\partial R_2}\Delta R_2 = \frac{(R_1+R_2)R_2 - R_1R_2(1)}{(R_1+R_2)^2}\Delta R_1 + \frac{(R_1+R_2)R_1 - R_1R_2(1)}{(R_1+R_2)^2}\Delta R_2$$

$$= \frac{(R_2)^2}{(R_1+R_2)^2}\Delta R_1 + \frac{(R_1)^2}{(R_1+R_2)^2}\Delta R_2 = \frac{6^2}{(4+6)^2}.03 + \frac{4^2}{(4+6)^2}.02 = \frac{1.08+0.32}{100} = 0.014\Omega$$

16. Find the point(s) on the surface $z^2 = 46 - 4x - 2y$ which are closest to the origin.
HINT: Explain why you can minimize the square of the distance instead of the distance.
Use the Second Derivative Test to check it is a local minimum.

SOLUTION: We need to minimize the distance from the point $\left(x, y, \pm\sqrt{46-4x-2y}\right)$ to the origin.
We can minimize the square of the distance because as the distance decreases, so does it's square.
So we minimize $f = x^2 + y^2 + z^2 = x^2 + y^2 + 46 - 4x - 2y$

$f_x = 2x - 4 = 0 \quad \Rightarrow \quad x = 2$

$f_y = 2y - 2 = 0 \quad \Rightarrow \quad y = 1 \quad \Rightarrow \quad z = \pm\sqrt{46-4x-2y} = \pm\sqrt{36} = \pm 6$

So the points are $(2,1,6)$ and $(2,1,-6)$

$f_{xx} = 2 > 0 \quad f_{yy} = 2 \quad f_{xy} = 0 \quad D = f_{xx}f_{yy} - f_{xy}^{\,2} = 4 > 0$ local minimum