1-8	/48
9	/12
10	/20
11	/20
Total	/100

Name__________________________ Section_____________

MATH 253 Exam 2 Fall 2012

Sections 201-202 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. Compute $\int_0^3 \int_y^3 4x^2\,dx\,dy$.

a. 81 Correct Choice

b. 72

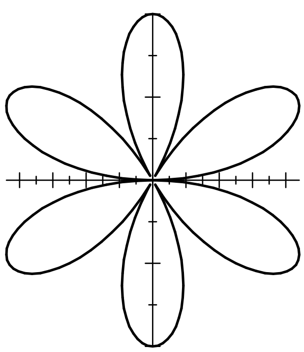
c. 60

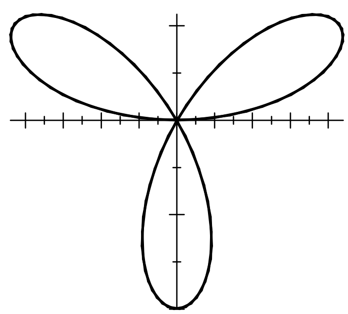
d. 48

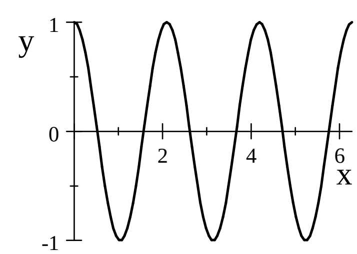
e. 32

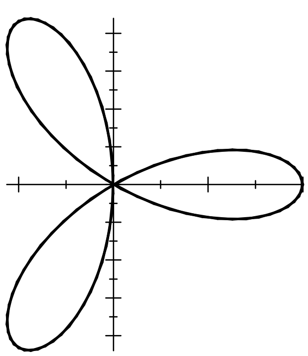
SOLUTION: $\int_0^3 \int_y^3 4x^2\,dx\,dy = \int_0^3 \left[4\frac{x^3}{3}\right]_{x=y}^3 dy = \int_0^3 36 - 4\frac{y^3}{3}\,dy = \left[36y - \frac{y^4}{3}\right]_0^3 = 108 - 27 = 81$

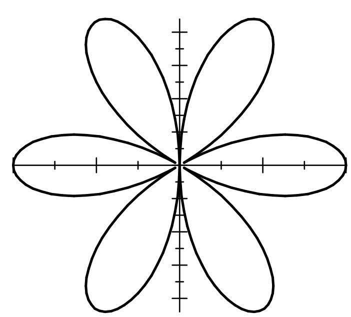
2. Which of the following is the polar plot of $r = \cos(3\theta)$?

a. 

b. 

c. 

d.  ←Correct Choice

e. 

SOLUTION: (c) is the rectangular plot of $r = \cos(3\theta)$. (d) is its polar plot because there are 3 positive loops and 3 negative loops which retrace the positive loops with $r = 1$ when $\theta = 0$.

3. Find the mass of a triangular plate whose vertices are $(0,0)$, $(1,0)$ and $(1,3)$, if the density is $\rho = 2x$.

a. 1

b. 2 Correct Choice

c. 3

d. 4

e. 5

SOLUTION: $M = \int\int \rho\, dA = \int_0^1 \int_0^{3x} 2x\, dy\, dx = \int_0^1 [2xy]_{y=0}^{3x}\, dx = \int_0^1 6x^2\, dx = [2x^3]_0^1 = 2$

4. Find the x-component of the center of mass of a triangular plate whose vertices are $(0,0)$, $(1,0)$ and $(1,3)$, if the density is $\rho = 2x$.

a. $\frac{1}{4}$

b. $\frac{1}{2}$

c. $\frac{3}{4}$ Correct Choice

d. $\frac{3}{2}$

e. 3

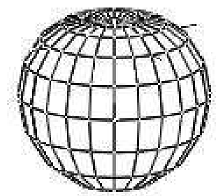
SOLUTION: $M_y = \int\int x\rho\, dA = \int_0^1 \int_0^{3x} 2x^2\, dy\, dx = \int_0^1 [2x^2 y]_{y=0}^{3x}\, dx = \int_0^1 6x^3\, dx = \left[\frac{3}{2}x^4\right]_0^1 = \frac{3}{2}$

$\bar{x} = \frac{M_y}{M} = \frac{3}{2}\frac{1}{2} = \frac{3}{4}$

5. The surface of an apple is given in spherical coordinates by

$$\rho = 3 - 3\cos\varphi$$

Its volume is given by the integral:



a. $V = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{3-3\cos\varphi} 1\, d\rho\, d\varphi\, d\theta$

b. $V = \int_0^{2\pi} \int_0^{\pi} \int_0^{3-3\cos\varphi} 1\, d\rho\, d\varphi\, d\theta$

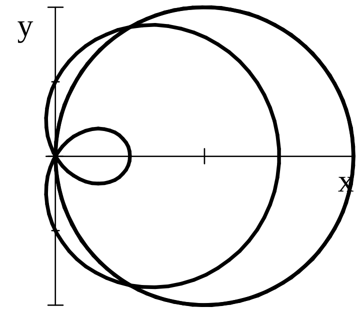
c. $V = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{3-3\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

d. $V = \int_0^{2\pi} \int_0^{\pi} \int_0^{3-3\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$ Correct Choice

e. $V = \int_0^{2\pi} \int_0^{\pi} \int_0^{1} (3 - 3\cos\varphi)\rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

SOLUTION: $V = \iiint dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{3-3\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

6. Find the area inside the circle $r = 4\cos\theta$ and outside the limacon $r = 1 + 2\cos\theta$.



a. $4\pi - \sqrt{3}$

b. $\frac{5\pi}{3} + \frac{\sqrt{3}}{2}$

c. $2\pi + \frac{\sqrt{3}}{2}$

d. $\frac{5\pi}{3} - \frac{\sqrt{3}}{2}$ Correct Choice

e. $2\pi - \frac{\sqrt{3}}{2}$

SOLUTION: Find the angles of intersection: $4\cos\theta = 1 + 2\cos\theta \qquad \cos\theta = \frac{1}{2} \qquad \theta = \pm\frac{\pi}{3}$

$$A = \int\int 1\,dA = 2\int_0^{\pi/3}\int_{1+2\cos\theta}^{4\cos\theta} 1\,r\,dr\,d\theta = \int_0^{\pi/3}[r^2]_{1+2\cos\theta}^{4\cos\theta}\,d\theta = \int_{-\pi/3}^{\pi/3} 16\cos^2\theta - (1+2\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi/3} 16\cos^2\theta - (1 + 4\cos\theta + 4\cos^2\theta)\,d\theta = \int_0^{\pi/3} 6(1+\cos(2\theta)) - 1 - 4\cos\theta\,d\theta$$

$$= \int_0^{\pi/3} 5 + 6\cos(2\theta) - 4\cos\theta\,d\theta = [5\theta + 3\sin(2\theta) - 4\sin\theta]_0^{\pi/3} = \frac{5\pi}{3} + 3\sin\frac{2\pi}{3} - 4\sin\frac{\pi}{3}$$

$$= \frac{5\pi}{3} + \frac{3\sqrt{3}}{2} - \frac{4\sqrt{3}}{2} = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$$

7. Hyperbolic coordinates in quadrant I are given by $u = \sqrt{\frac{y}{x}}$ and $v = \sqrt{yx}$.
So the area element is $dA = dx\,dy =$

a. $-2\frac{v}{u}\,du\,dv$

b. $2\frac{v}{u}\,du\,dv$ Correct Choice

c. $-2\frac{u}{v}\,du\,dv$

d. $2\frac{u}{v}\,du\,dv$

e. $2\frac{u^2}{v^2}\,du\,dv$

SOLUTION: $uv = y \qquad \frac{v}{u} = x \qquad x = \frac{v}{u} \qquad y = uv$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{-v}{u^2} & v \\ \frac{1}{u} & u \end{vmatrix} = \frac{-v}{u} - \frac{v}{u} = -2\frac{v}{u} \qquad J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 2\frac{v}{u} \qquad dA = 2\frac{v}{u}\,du\,dv$$

8. If $f = \sin(x - y)$, then $\vec{\nabla} \cdot \vec{\nabla} f =$

a. $2\sin(x - y)$
b. $-2\sin(x - y)$ Correct Choice
c. $2\cos(x - y)$
d. $-2\cos(x - y)$
e. 0

SOLUTION: $\vec{\nabla} f = (\cos(x - y), -\cos(x - y))$ $\vec{\nabla} \cdot \vec{\nabla} f = -\sin(x - y) - \sin(x - y) = -2\sin(x - y)$

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (12 points) Determine whether or not each of these limits exists. If it exists, find its value.

a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y^2}{x^6 + 3y^3}$

SOLUTION: Straight line approaches: $y = mx$

$$\lim_{\substack{y=mx \\ x\to 0}} \frac{3x^2y^2}{x^6 + 3y^3} = \lim_{x\to 0} \frac{3x^2m^2x^2}{x^6 + 3m^3x^3} = \lim_{x\to 0} \frac{3m^2x}{x^3 + 3m^3} = \frac{0}{3m^3} = 0$$

Quadratic approaches: $y = mx^2$

$$\lim_{\substack{y=mx^2 \\ x\to 0}} \frac{3x^2y^2}{x^6 + 3y^3} = \lim_{x\to 0} \frac{3x^2m^2x^4}{x^6 + 3m^3x^6} = \lim_{x\to 0} \frac{3m^2}{1 + 3m^3} \ne 0 \qquad \text{if } m \ne 0.$$

Limit does not exist because these are different.

b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^2}$

SOLUTION: Switch to polar: $x = r\cos\theta$ $y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} = \lim_{\substack{r\to 0 \\ \theta \text{ arbitrary}}} \frac{r\cos\theta\, r^2\sin^2\theta}{r^2} = \lim_{\substack{r\to 0 \\ \theta \text{ arbitrary}}} r\cos\theta\sin^2\theta = 0$$

because $r \to 0$ while $\cos\theta\sin^2\theta$ is bounded: $-1 \le \cos\theta\sin^2\theta \le 1$.

10. (20 points) Compute $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = (yz, -xz, z^2)$ over the cone $z = 9 - \sqrt{x^2 + y^2}$ for $z \geq 5$ oriented down and in.

Note: The cone may be parametrized as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 9 - r)$.

SOLUTION: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz & -xz & z^2 \end{vmatrix} = \hat{\imath}(0 - -x) - \hat{\jmath}(0 - y) + \hat{k}(-z - z) = (x, y, -2z)$

$\left(\vec{\nabla} \times \vec{F}\right)\left(\vec{R}(r,\theta)\right) = (r\cos\theta, r\sin\theta, -2(9 - r)) = (r\cos\theta, r\sin\theta, 2r - 18)$

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\quad \cos\theta & \sin\theta & -1) \\ \vec{e}_\theta = & (-r\sin\theta & r\cos\theta & 0) \end{array}$$

$\vec{N} = \hat{\imath}(0 - -r\cos\theta) - \hat{\jmath}(0 - r\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (r\cos\theta, r\sin\theta, r)$ up and out

Reverse $\vec{N} = (-r\cos\theta, -r\sin\theta, -r)$ now down and in

$\vec{\nabla} \times \vec{F} \cdot \vec{N} = -r^2\cos^2\theta - r^2\sin^2\theta - r(2r - 18) = -3r^2 + 18r \qquad 9 - r = 5 \qquad r = 4$

$\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^4 -3r^2 + 18r\, dr\, d\theta = 2\pi[-r^3 + 9r^2]_0^4 = 2\pi(-64 + 144) = 160\pi$

11. (20 points) Compute $\iiint \vec{\nabla} \cdot \vec{F}\, dV$ for the vector field $\vec{F} = (x^3, y^3, x^2z + y^2z)$ over the solid region below the paraboloid $z = 9 - x^2 - y^2$ and above the plane $z = 5$.

SOLUTION: $\vec{\nabla} \cdot \vec{F} = 3x^2 + 3y^2 + x^2 + y^2 = 4(x^2 + y^2) = 4r^2 \qquad 5 = 9 - r^2 \qquad r = 2$

$\iiint \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi}\int_0^2\int_5^{9-r^2} 4r^2\, r\, dz\, dr\, d\theta = 2\pi\int_0^2 [4r^3z]_{z=5}^{9-r^2}\, dr = 2\pi\int_0^2 4r^3(4 - r^2)\, dr$

$= 8\pi\left[r^4 - \frac{r^6}{6}\right]_0^2 = 8\pi\left(16 - \frac{32}{3}\right) = 128\pi\left(1 - \frac{2}{3}\right) = \frac{128\pi}{3}$