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MATH 253H Final Exam Fall 2012

Sections 201-202 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Points A, B, C and D are the vertices of a parallelogram traversed in order. If $A = (2,4,1)$, $B = (3,-2,0)$ and $D = (1,3,-2)$, then $C =$

a. $(2,-3,-3)$ Correct Choice

b. $(4,-1,3)$

c. $(0,9,-1)$

d. $(6,5,-1)$

e. $\left(4,\frac{9}{2},0\right)$

SOLUTION: $\overrightarrow{AD} = D - A = (1,3,-2) - (2,4,1) = (-1,-1,-3)$
$C = B + \overrightarrow{AD} = (3,-2,0) + (-1,-1,-3) = (2,-3,-3)$

2. Which vector is perpendicular to the surface $x^2z^3 + y^3z^2 = 1$ at the point $(3,-2,1)$?

a. $(12,-24,43)$

b. $(6,-12,43)$

c. $(6,12,43)$

d. $(6,-12,11)$

e. $(-12,-24,-22)$ Correct Choice

SOLUTION: $f = x^2z^3 + y^3z^2$ $\quad \vec{\nabla}f = (2xz^3, 3y^2z^2, 3x^2z^2 + 2y^3z)$ $\quad \vec{\nabla}f\Big|_{(3,-2,1)} = (6,12,11)$
$(-12,-24,-22) = -2(6,12,11)$

3. Find the point on the elliptic paraboloid $\vec{R}(t,\theta) = (3t\cos\theta, 2t\sin\theta, 1+t^2)$ where a unit normal is $\hat{N} = \left(\frac{-2\sqrt{3}}{5}, \frac{-2}{5}, \frac{3}{5}\right)$.

a. $\left(\frac{3}{2}, \sqrt{3}, 2\right)$
b. $\left(-\frac{3}{2}\sqrt{3}, -1, 2\right)$
c. $\left(3, 2\sqrt{3}, 5\right)$
d. $\left(3\sqrt{3}, 2, 5\right)$ Correct Choice
e. $\left(-3, -2\sqrt{3}, 5\right)$

SOLUTION:

$$\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_t = & (3\cos\theta, & 2\sin\theta, & 2t) \\ \vec{e}_\theta = & (-3t\sin\theta, & 2t\cos\theta, & 0) \end{vmatrix} \qquad \vec{N} = (-4t^2\cos\theta, -4t^2\sin\theta, 6t)$$

$$\left|\vec{N}\right| = \sqrt{16t^4 + 36t^2} = 2t\sqrt{4t^2+9} \qquad \hat{N} = \left(\frac{-2t\cos\theta}{\sqrt{4t^2+9}}, \frac{-2t\sin\theta}{\sqrt{4t^2+9}}, \frac{3}{\sqrt{4t^2+9}}\right) = \left(\frac{-2\sqrt{3}}{5}, \frac{-2}{5}, \frac{3}{5}\right)$$

$$\sqrt{4t^2+9} = 5 \qquad t = 2 \qquad -4\cos\theta = -2\sqrt{3} \qquad \cos\theta = \frac{\sqrt{3}}{2} \qquad \theta = \frac{\pi}{6}$$

$$\vec{R}\left(2, \frac{\pi}{6}\right) = \left(3\cdot 2\cos\frac{\pi}{6}, 2\cdot 2\sin\frac{\pi}{6}, 1+2^2\right) = \left(6\frac{\sqrt{3}}{2}, 4\frac{1}{2}, 5\right) = \left(3\sqrt{3}, 2, 5\right)$$

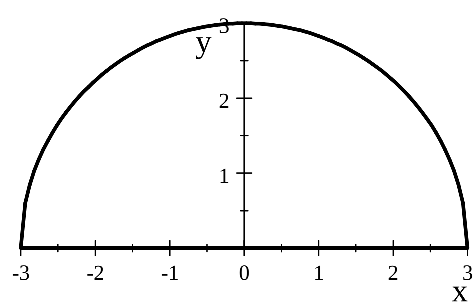
4. Compute $\displaystyle\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \frac{1}{1+x^2+y^2}\,dx\,dy$

HINT: Plot the region of integration and convert to polar coordinates.

a. $\frac{\pi}{2}\ln 10$
b. $\pi\ln 10$ Correct Choice
c. $\frac{\pi}{2}\arctan 3$
d. $\pi\arctan 3$
e. $\pi\arctan 10$

SOLUTION:

$$0 \le y \le 3$$
$$-\sqrt{9-y^2} \le x \le \sqrt{9-y^2}$$
or
$$0 \le r \le 3$$
$$0 \le \theta \le \pi$$



$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \frac{1}{1+x^2+y^2}\,dx\,dy = \int_0^\pi \int_0^3 \frac{r}{1+r^2}\,dr\,d\theta = \left[\frac{\pi}{2}\ln(1+r^2)\right]_0^3 = \frac{\pi}{2}\ln 10$$

5. Find the mass of a wire in the shape of the curve $\vec{r}(t) = \left(e^t, \sqrt{2}\,t, e^{-t}\right)$ for $-1 \le t \le 1$ if the density is $\rho = x$.

a. $\dfrac{e^2}{2} + \dfrac{e^{-2}}{2}$

b. $\dfrac{e^2}{2} - \dfrac{e^{-2}}{2}$

c. $\dfrac{e^2}{2} - \dfrac{e^{-2}}{2} + 2$ Correct Choice

d. $e^2 - e^{-2}$

e. $e^2 - e^{-2} + 2$

SOLUTION: $\vec{v} = \left(e^t, \sqrt{2}, -e^{-t}\right)$ $|\vec{v}| = \sqrt{e^{2t} + 2 + e^{-2t}} = e^t + e^{-t}$

$$M = \int \rho\, ds = \int_{-1}^{1} x|\vec{v}|\,dt = \int_{-1}^{1} e^t(e^t + e^{-t})\,dt = \int_{-1}^{1} (e^{2t} + 1)\,dt = \left[\frac{e^{2t}}{2} + t\right]_{-1}^{1}$$
$$= \left(\frac{e^2}{2} + 1\right) - \left(\frac{e^{-2}}{2} - 1\right) = \frac{e^2}{2} - \frac{e^{-2}}{2} + 2$$

6. Find the plane tangent to graph of $z = x\cos y + \sin y$ at $(2, \pi)$.
What is the z-intercept?

a. $-4 + \pi$

b. $4 + \pi$

c. $-4 - \pi$

d. $4 - \pi$

e. π Correct Choice

SOLUTION: $f(x,y) = x\cos y + \sin y$ $f(2,\pi) = 2\cos\pi + \sin\pi = -2$

$f_x(x,y) = \cos y$ $f_x(2,\pi) = \cos\pi = -1$

$f_y(x,y) = -x\sin y + \cos y$ $f_y(2,\pi) = -2\sin\pi + \cos\pi = -1$

$z = f(2,\pi) + f_x(2,\pi)(x-2) + f_y(2,\pi)(y-\pi)$

tan plane: $z = -2 - 1(x-2) - 1(y-\pi)$

z-intercept: $x = y = 0$ $z = -2 - 1(-2) - 1(-\pi) = \pi$

7. Let $L = \lim_{(x,y)\to(0,0)} \dfrac{e^{(x^2+y^2)} - 1}{x^2 + y^2}$

a. L does not exist by looking at the paths $y = x$ and $y = -x$.
b. L exists and $L = 1$ by looking at the paths $y = mx$.
c. L does not exist by looking at polar coordinates.
d. L exists and $L = 1$ by looking at polar coordinates. Correct Choice
e. L exists and $L = 0$ by looking at polar coordinates.

SOLUTION: Along $y = mx$, we have $L = \lim_{x\to0} \dfrac{e^{(1+m^2)x^2} - 1}{(1+m^2)x^2} \overset{\text{l'H}}{=} \lim_{x\to0} \dfrac{e^{(1+m^2)x^2}(1+m^2)2x}{(1+m^2)2x} = 1$,
for all m including 1 and -1 which proves nothing.
In polar coordinates, $L = \lim_{r\to0} \dfrac{e^{r^2} - 1}{r^2} \overset{\text{l'H}}{=} \lim_{r\to0} \dfrac{e^{r^2}2r}{2r} = 1$, which proves the limit exists and $= 1$.

8. Compute $\oint \vec{F} \cdot d\vec{s}$ for $\vec{F} = (y^2, 4xy)$ along the piece of the parabola $y = x^2$ from $(-2,4)$ to $(2,4)$ followed by the line segment from $(2,4)$ back to $(-2,4)$.
HINT: Use Green's Theorem.

a. $\dfrac{256}{5}$ Correct Choice
b. $\dfrac{768}{5}$
c. $\dfrac{64}{3}$
d. $\dfrac{128}{3}$
e. 0

SOLUTION: $\oint \vec{F} \cdot d\vec{s} = \oint P\,dx + Q\,dy$ with $P = y^2$ and $Q = 4xy$. By Green's Theorem,
$\oint \vec{F} \cdot d\vec{s} = \int_{-2}^{2} \int_{x^2}^{4} (\partial_x Q - \partial_y P)\,dy\,dx = \int_{-2}^{2} \int_{x^2}^{4} (4y - 2y)\,dy\,dx = \int_{-2}^{2} [y^2]_{x^2}^{4}\,dx$
$= \int_{-2}^{2} (16 - x^4)\,dx = \left[16x - \dfrac{x^5}{5}\right]_{-2}^{2} = 2\left(32 - \dfrac{32}{5}\right) = \dfrac{256}{5}$

9. Compute $\int \vec{F} \cdot d\vec{s}$ for $\vec{F} = (4xy^2, 4x^2y)$ along the line segment from $(1,2)$ to $(3,1)$.
HINT: Find a scalar potential.

a. 4
b. 10 Correct Choice
c. 20
d. 24
e. 26

SOLUTION: $\vec{F} = (4xy^2, 4x^2y) = \vec{\nabla} f$ for $f = 2x^2y^2$ since $\partial_x f = 4xy^2$ and $\partial_y f = 4x^2y$
By the F.T.C.C. $\int \vec{F} \cdot d\vec{s} = \int_{(1,2)}^{(3,1)} \vec{\nabla} f \cdot d\vec{s} = f(3,1) - f(1,2) = 2(3)^2(1)^2 - 2(1)^2(2)^2 = 10$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (15 points) A 166 cm piece of wire is cut into 3 pieces of lengths a, b and c.
The piece of length a is folded into a square of side $s = \frac{a}{4}$.
The piece of length b is folded into a rectangle of length $L_1 = \frac{b}{3}$ and width $W_1 = \frac{b}{6}$.
The piece of length c is folded into a rectangle of length $L_2 = \frac{3c}{8}$ and width $W_2 = \frac{c}{8}$.
Find a, b and c so that the total area is a minimum.
What is the total area?

Minimize the total area: $A = s^2 + L_1 W_1 + L_2 W_2 = \frac{a^2}{16} + \frac{b^2}{18} + \frac{3c^2}{64}$

subject to the constraint: $g = a + b + c = 166$

METHOD 1: Lagrange Multipliers:

$\vec{\nabla} A = \left(\frac{a}{8}, \frac{b}{9}, \frac{3c}{32}\right) \qquad \vec{\nabla} g = (1,1,1)$

$\vec{\nabla} A = \lambda \vec{\nabla} g \quad \Rightarrow \quad \frac{a}{8} = \lambda \quad \frac{b}{9} = \lambda \quad \frac{3c}{32} = \lambda \quad \Rightarrow \quad \frac{a}{8} = \frac{b}{9} = \frac{3c}{32}$

$a = \frac{3c}{4} \qquad b = \frac{27c}{32}$

Constraint: $166 = a + b + c = \frac{3c}{4} + \frac{27c}{32} + c = \frac{83}{32}c \quad \Rightarrow \quad c = 64$

$a = \frac{3c}{4} = 48 \qquad b = \frac{27c}{32} = 54$

$A = \frac{48^2}{16} + \frac{54^2}{18} + \frac{3 \cdot 64^2}{64} = 498$

METHOD 2: Eliminate a Variable:

$c = 166 - a - b$

$A = \frac{a^2}{16} + \frac{b^2}{18} + \frac{3(166 - a - b)^2}{64}$

$A_a = \frac{a}{8} - \frac{3(166 - a - b)}{32} = \frac{7}{32}a + \frac{3}{32}b - \frac{249}{16} = 0 \quad \Rightarrow \quad 7a + 3b = 498 \quad (1)$

$A_b = \frac{b}{9} - \frac{3(166 - a - b)}{32} = \frac{3}{32}a + \frac{59}{288}b - \frac{249}{16} = 0 \quad \Rightarrow \quad 3a + \frac{59}{9}b = 498 \quad (2)$

$3(1) - 7(2)$: $9b - 7 \cdot \frac{59}{9}b = -4 \cdot 498 \quad \Rightarrow \quad -\frac{332}{9}b = -1992 \quad \Rightarrow \quad b = 54$

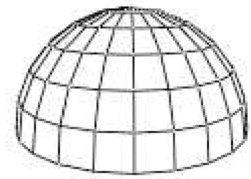
$\frac{59}{9}(1) - 3(2)$: $\frac{59}{9} \cdot 7a - 9a = \left(\frac{59}{9} - 3\right)498 \quad \Rightarrow \quad \frac{332}{9}a = \frac{5312}{3} \quad \Rightarrow \quad a = 48$

$c = 166 - a - b = 166 - 48 - 54 = 64$

$A = \frac{48^2}{16} + \frac{54^2}{18} + \frac{3 \cdot 64^2}{64} = 498$

11. (25 points) Verify Gauss' Theorem $\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_{\partial V} \vec{F} \cdot d\vec{S}$

for the vector field $\vec{F} = (xz, yz, x^2 + y^2)$ and the solid hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$.



Be careful with orientations. Use the following steps:

First the Left Hand Side:

a. Compute the divergence:

$\vec{\nabla} \cdot \vec{F} = z + z + 0 = 2z$

b. Express the divergence and the volume element in the appropriate coordinate system:

$\vec{\nabla} \cdot \vec{F} = 2\rho\cos\varphi \qquad dV = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

c. Compute the left hand side:

$$\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 2\rho\cos\varphi\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\int_0^{\pi/2}\cos\varphi\sin\varphi\, d\varphi \int_0^2 2\rho^3\, d\rho$$

$$= 2\pi\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/2}\left[\frac{\rho^4}{2}\right]_0^2 = 2\pi\left(\frac{1}{2}\right)(8) = 8\pi$$

Second the Right Hand Side:

The boundary surface consists of a hemisphere H and a disk D with appropriate orientations.

d. Parametrize the disk D:

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$

e. Compute the tangent vectors:

$\vec{e}_r = (\cos\theta, \quad \sin\theta, \quad 0)$

$\vec{e}_\theta = (-r\sin\theta, \quad r\cos\theta, \quad 0)$

f. Compute the normal vector:

$\vec{N} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (0,0,r)$

This is up. Need down. Reverse: $\vec{N} = (0,0,-r)$

g. Evaluate $\vec{F} = (xz, yz, x^2 + y^2)$ on the disk:

$\vec{F}\Big|_{\vec{R}(r,\theta)} = (0,0,r^2)$

h. Compute the dot product:

$\vec{F} \cdot \vec{N} = -r^3$

i. Compute the flux through D:

$$\iint\limits_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 -r^3\, dr\, d\theta = 2\pi\left[-\frac{r^4}{4}\right]_0^2 = -8\pi$$

j. Parametrize the hemisphere H:

$$\vec{R}(\varphi,\theta) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$$

k. Compute the tangent vectors:

$$\vec{e}_\varphi = (2\cos\varphi\cos\theta, \quad 2\cos\varphi\sin\theta, \quad -2\sin\varphi)$$

$$\vec{e}_\theta = (-2\sin\varphi\sin\theta, \quad 2\sin\varphi\cos\theta, \quad 0)$$

l. Compute the normal vector:

$$\vec{N} = \hat{\imath}(4\sin^2\varphi\cos\theta) - \hat{\jmath}(-4\sin^2\varphi\sin\theta) + \hat{k}(4\sin\varphi\cos\varphi\sin^2\theta + 4\sin\varphi\cos\varphi\cos^2\theta)$$

$$= (4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi)$$

This is oriented correctly as up and out.

m. Evaluate $\vec{F} = (xz, yz, x^2 + y^2)$ on the hemisphere:

$$\vec{F}\Big|_{\vec{R}(\theta,\varphi)} = (4\sin\varphi\cos\varphi\cos\theta, 4\sin\varphi\cos\varphi\sin\theta, 4\sin^2\varphi)$$

n. Compute the dot product:

$$\vec{F}\cdot\vec{N} = 16\sin^3\varphi\cos\varphi\cos^2\theta + 16\sin^3\varphi\cos\varphi\sin^2\theta + 16\sin^3\varphi\cos\varphi$$

$$= 16\sin^3\varphi\cos\varphi + 16\sin^3\varphi\cos\varphi = 32\sin^3\varphi\cos\varphi$$

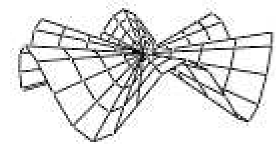
o. Compute the flux through H:

$$\iint_C \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^{\pi/2} 32\sin^3\varphi\cos\varphi\, d\varphi\, d\theta = 2\pi[8\sin^4\varphi]_0^{\pi/2} = 16\pi$$

p. Compute the **TOTAL** right hand side:

$$\iint_{\partial V} \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot d\vec{S} + \iint_H \vec{F}\cdot d\vec{S} = -8\pi + 16\pi = 8\pi$$ which agrees with (c).

12. (15 points) Compute $\int\int \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ for $\vec{F} = (-y, x, z)$



over the "clam shell" surface parametrized by

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r\sin(6\theta))$$

for $r \le 2$ oriented upward.

HINTS: Use Stokes Theorem.

What is the value of r on the boundary?

SOLUTION:

Boundary: $r = 2$ $\quad \vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 2\sin(6\theta))$ $\quad \vec{v} = (-2\sin\theta, 2\cos\theta, 12\cos(6\theta))$

$\vec{F}(\vec{r}(\theta)) = (-2\sin\theta, 2\cos\theta, 2\sin(6\theta))$

$\vec{F} \cdot \vec{v} = 4\sin^2\theta + 4\cos^2\theta + 24\sin(6\theta)\cos(6\theta) = 4 + 24\sin(6\theta)\cos(6\theta)$

$$\int\int \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 4 + 24\sin(6\theta)\cos(6\theta)\, d\theta = \Big[4\theta + 2\sin^2(6\theta) \Big]_0^{2\pi} = 8\pi$$