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MATH 253 Exam 2 Fall 2014

Sections 201,202 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Compute $\int_0^\pi \int_0^\pi \int_0^\varphi \rho^2 \, d\rho \, d\theta \, d\varphi$.

a. $\frac{1}{6}\pi^5$

b. $\frac{1}{12}\pi^5$ Correct Choice

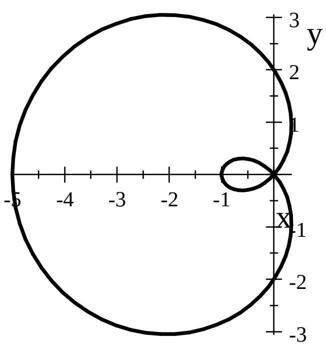
c. $\frac{1}{20}\pi^5$

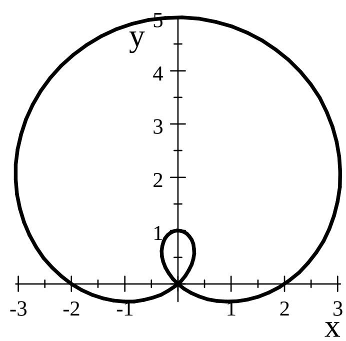
d. $\frac{1}{6}\pi^4$

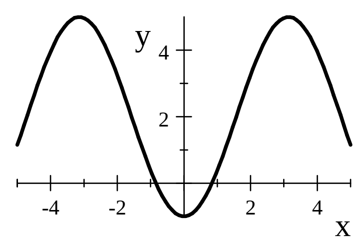
e. $\frac{1}{12}\pi^4$

Solution: $\int_0^\pi \int_0^\pi \int_0^\varphi \rho^2 \, d\rho \, d\theta \, d\varphi = \int_0^\pi \int_0^\pi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\varphi} d\theta \, d\varphi = \pi \int_0^\pi \frac{\varphi^3}{3} d\varphi = \left[\pi \frac{\varphi^4}{12} \right]_0^\pi = \frac{\pi^5}{12}$

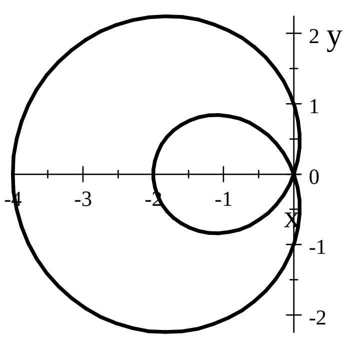
2. Which of the following is the polar plot of $r = 2 - 3\cos(\theta)$?

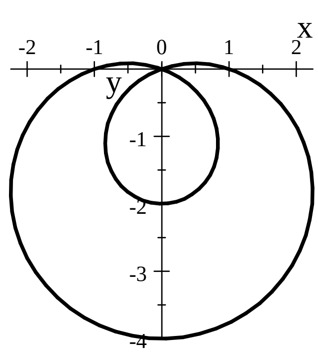
a. 

b. 

c. 

Correct Choice ⇑

d. 

e. 

Solution: c is the rectangular plot not the polar plot. Notice when $\theta = 0$, $r = -1$, which starts the inner loop at $x = -1$.

3. A plate fills the region between the graphs of $x = |y|$ and $x = 4$. Find its mass if its surface density is $\rho = x^2$.

a. $\frac{256}{3}$
b. $\frac{128}{3}$
c. 256
d. 128 Correct Choice
e. 64

Solution: $M = \int\int \rho\, dA = \int_0^4 \int_{-x}^{x} x^2\, dy\, dx = \int_0^4 [x^2 y]_{y=-x}^{x}\, dx = \int_0^4 (x^3 - -x^3)\, dx = \left[2\frac{x^4}{4}\right]_0^4 = 128$

4. A plate fills the region between the graphs of $x = |y|$ and $x = 4$. Find the x-component of its center of mass if its surface density is $\rho = x^2$.

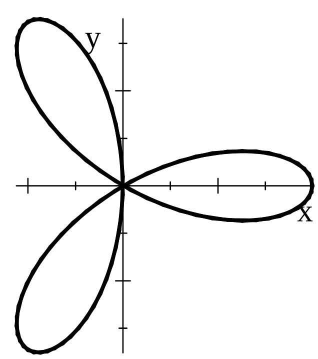
a. $\frac{16}{5}$ Correct Choice
b. $\frac{32}{5}$
c. $\frac{512}{5}$
d. $\frac{2048}{5}$
e. $\frac{5}{2048}$

Solution: $M_y = \int\int x\rho\, dA = \int_0^4 \int_{-x}^{x} x x^2\, dy\, dx = \int_0^4 [x^3 y]_{y=-x}^{x}\, dx = \int_0^4 (x^4 - -x^4)\, dx = \left[2\frac{x^5}{5}\right]_0^4 = \frac{2048}{5}$

$\bar{x} = \frac{M_y}{M} = \frac{2048}{5} \cdot \frac{1}{128} = \frac{16}{5}$

5. Find the area of one leaf of the rose

$r = 2\cos 3\theta.$



a. $\frac{\pi}{12}$
b. $\frac{\pi}{6}$
c. $\frac{\pi}{3}$ Correct Choice
d. $\frac{2\pi}{3}$
e. $\frac{4\pi}{3}$

Solution: Find where the radius is zero: $2\cos 3\theta = 0 \quad \cos 3\theta = 0 \quad 3\theta = \pm\frac{\pi}{2} \quad \theta = \pm\frac{\pi}{6}$

$A = \int\int 1\, dA = \int_{-\pi/6}^{\pi/6} \int_0^{2\cos 3\theta} r\, dr\, d\theta = \int_{-\pi/6}^{\pi/6} \left[\frac{r^2}{2}\right]_{r=0}^{2\cos 3\theta} d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (2\cos 3\theta)^2\, d\theta$

$= 2\int_{-\pi/6}^{\pi/6} \cos^2 3\theta\, d\theta = 2\int_{-\pi/6}^{\pi/6} \frac{1 + \cos 6\theta}{2}\, d\theta = \left[\theta + \frac{\sin 6\theta}{6}\right]_{-\pi/6}^{\pi/6} = \frac{\pi}{3}$

6. Set up, but do not compute, the integral $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV$ over the solid region between the paraboloid $z = 2x^2 + 2y^2$ and the plane $z = 8$ where $\vec{F} = (x^3, y^3, z(x^2+y^2))$.

a. $\int_0^{\pi} \int_0^2 \int_{2r^2}^8 4r^2\, dz\, dr\, d\theta$

b. $\int_0^{2\pi} \int_0^2 \int_{2r^2}^8 4r^2\, dz\, dr\, d\theta$

c. $\int_0^{2\pi} \int_0^2 \int_{2r^2}^8 4r^3\, dz\, dr\, d\theta$ Correct Choice

d. $\int_0^{2\pi} \int_0^8 \int_0^{z/2} 4r^2\, dr\, dz\, d\theta$

e. $\int_0^{2\pi} \int_0^8 \int_0^{z/2} 4r^3\, dr\, dz\, d\theta$

Solution: Take the divergence in rectangluar coordinates, then use cylindrical coordinates.

$\vec{\nabla} \cdot \vec{F} = 3x^2 + 3y^2 + (x^2+y^2) = 4(x^2+y^2) = 4r^2$

$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^2 \int_{2r^2}^8 4r^2\, r\, dz\, dr\, d\theta$

7. Find the average value of $f = z$ on the solid hemisphere $0 \le z \le \sqrt{9 - x^2 - y^2}$.

Note: The average value of a function on a solid is $f_{\text{ave}} = \dfrac{1}{V} \iiint_V f\, dV$.

a. $\dfrac{3\pi}{8}$

b. $\dfrac{\pi}{2}$

c. $\dfrac{3}{2}$

d. $\dfrac{1}{2}$

e. $\dfrac{9}{8}$ Correct Choice

Solution: $V = \dfrac{1}{2}\dfrac{4}{3}\pi r^3 = \dfrac{2}{3}\pi 3^3 = 18\pi$. In spherical coordinates, $f = z = \rho\cos\varphi$.

$\iiint_V f\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 \rho\cos\varphi\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi \left[\dfrac{\sin^2\varphi}{2} \right]_0^{\pi/2} \left[\dfrac{\rho^4}{4} \right]_0^3 = \dfrac{3^4\pi}{4}$

$f_{\text{ave}} = \dfrac{1}{V} \iiint_V f\, dV = \dfrac{1}{18\pi} \cdot \dfrac{3^4\pi}{4} = \dfrac{9}{8}$

8. Compute the line integral $\int \vec{\nabla} \times \vec{F} \cdot d\vec{s}$ counterclockwise around one loop of the helix $\vec{r}(\theta) = (4\cos\theta, 4\sin\theta, 3\theta)$ for the vector field $\vec{F} = (xz, yz, z^2)$.

Hint: Compute $\vec{\nabla} \times \vec{F}$ in rectangular coordinates before integrating.

a. 16π

b. 32π Correct Choice

c. 64π

d. 128π

e. 0

Solution: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xz & yz & z^2 \end{vmatrix} = \hat{\imath}(0 - y) - \hat{\jmath}(0 - x) + \hat{k}(0) = (-y, x, 0) = (-4\sin\theta, 4\cos\theta, 0)$

$\vec{v} = (-4\sin\theta, 4\cos\theta, 3)$ $\quad \int \vec{\nabla} \times \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{\nabla} \times \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 16\sin^2\theta + 16\cos^2\theta\, d\theta = \int_0^{2\pi} 16\, d\theta = 32\pi$

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (20 points) A rectangular solid box is sitting on the xy-plane with its upper 4 vertices on the ellipsoid $x^2 + 4y^2 + 9z^2 = 108$. Find the dimensions and volume of the largest such box.
Full credit for solving by Lagrange multipliers.
Half credit for solving by Eliminating a Variable.
50% extra credit for solving both ways.

Solution: Maximize: $V = 4xyz$. Note: $x \neq 0 \quad y \neq 0 \quad z \neq 0$ so the volume will not be zero.

Method 1: Lagrange Multipliers:

The constraint is $g = x^2 + 4y^2 + 9z^2 = 108$

$\vec{\nabla}V = \langle 4yz, 4xz, 4xy \rangle \qquad \vec{\nabla}g = \langle 2x, 8y, 18z \rangle \qquad \vec{\nabla}V = \lambda\vec{\nabla}g$

$4yz = \lambda 2x \qquad 4xz = \lambda 8y \qquad 4xy = \lambda 18z$

$\lambda = \dfrac{2yz}{x} = \dfrac{xz}{2y} = \dfrac{2xy}{9z} \quad \Rightarrow \quad 4y^2 = x^2 \qquad 9z^2 = x^2$

$g = x^2 + x^2 + x^2 = 108 \quad \Rightarrow \quad x^2 = 36 \quad x = 6$

$4y^2 = 36 \qquad y = 3 \qquad 9z^2 = 36 \qquad z = 2$

$L = 12, \qquad W = 6, \qquad H = 2, \qquad V = 4xyz = LWH = 4 \bullet 6 \bullet 3 \bullet 2 = 144$

Method 2: Eliminate a Variable:

$V = 4xyz = 4yz\sqrt{108 - 4y^2 - 9z^2}$

$$V_y = 4z\sqrt{108 - 4y^2 - 9z^2} + \frac{4yz(-8y)}{2\sqrt{108 - 4y^2 - 9z^2}} = 0 \qquad \times \frac{\sqrt{108 - 4y^2 - 9z^2}}{4z}$$

$$V_z = 4y\sqrt{108 - 4y^2 - 9z^2} + \frac{4yz(-18z)}{2\sqrt{108 - 4y^2 - 9z^2}} = 0 \qquad \times \frac{\sqrt{108 - 4y^2 - 9z^2}}{4y}$$

$(108 - 4y^2 - 9z^2) + y(-4y) = 0 \quad \Rightarrow \quad 108 - 8y^2 - 9z^2 = 0 \quad \Rightarrow \quad 216 - 16y^2 - 18z^2 = 0$

$(108 - 4y^2 - 9z^2) + z(-9z) = 0 \quad \Rightarrow \quad 108 - 4y^2 - 18z^2 = 0$

$108 - 12y^2 = 0 \qquad y = 3 \qquad 9z^2 = 108 - 8y^2 = 108 - 72 = 36 \qquad z = 2$

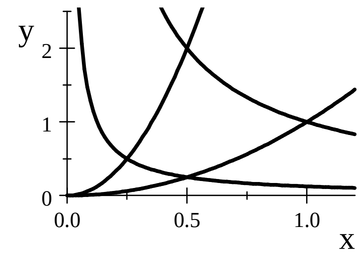
$x^2 = 108 - 4y^2 - 9z^2 = 108 - 36 - 36 = 36 \qquad x = 6$

$L = 12, \qquad W = 6, \qquad H = 2, \qquad V = 4xyz = LWH = 4 \bullet 6 \bullet 3 \bullet 2 = 144$

10. (20 points) Compute the integral $\iint y\,dA$ over the region in the first quadrant bounded by

$y = x^2$, $y = 8x^2$, $y = \frac{1}{x}$, and $y = \frac{1}{8x}$.

Use the following steps:



a. Define the curvilinear coordinates u and v by $y = u^3x^2$ and $y = \frac{v^3}{x}$.
Express the coordinate system as a position vector.

$$u^3x^2 = \frac{v^3}{x} \qquad x = \frac{v}{u} \qquad y = u^3x^2 = u^3\frac{v^2}{u^2} = uv^2$$
$$\vec{r}(u,v) = (x(u,v), y(u,v)) = \left(\frac{v}{u}, uv^2\right)$$

b. Find the coordinate tangent vectors:

$$\vec{e}_u = \frac{\partial\vec{r}}{\partial u} = \left(\frac{-v}{u^2}, v^2\right)$$

$$\vec{e}_v = \frac{\partial\vec{r}}{\partial v} = \left(\frac{1}{u}, 2uv\right)$$

c. Compute the Jacobian factor:

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{-2v^2}{u} - \frac{v^2}{u}\right| = \frac{3v^2}{u}$$

d. Compute the integral:

$$\iint y\,dA = \int_{1/2}^{1}\int_{1}^{2} uv^2\frac{3v^2}{u}\,du\,dv = \int_{1}^{2} 1\,du\int_{1/2}^{1} 3v^4\,dv = [u]_1^2\left[\frac{3v^5}{5}\right]_{1/2}^{1} = \frac{3}{5}\left(1 - \frac{1}{32}\right)$$
$$= \frac{93}{160}$$

11. (20 points) Compute the flux $\iint_H \vec{F} \cdot d\vec{S}$ of the vector field $\vec{F} = \left(\dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}}, \dfrac{z}{\sqrt{x^2+y^2}} \right)$ outward through the upper half of the sphere $x^2+y^2+z^2 = 9$. Use the following steps:

a. Parametrize the hemisphere:

$$\vec{R}(\theta,\varphi) = (3\sin\varphi\cos\theta, \quad 3\sin\varphi\sin\theta, \quad 3\cos\varphi)$$

b. Find the tangent vectors:

$$\vec{e}_\theta = (-3\sin\varphi\sin\theta, \quad 3\sin\varphi\cos\theta, \qquad 0)$$

$$\vec{e}_\varphi = (3\cos\varphi\cos\theta, \quad 3\cos\varphi\sin\theta, \quad -3\sin\varphi)$$

c. Find the normal vector:

$$\begin{aligned}\vec{N} &= \hat{\imath}(-9\sin^2\varphi\cos\theta) - \hat{\jmath}(9\sin^2\varphi\sin\theta) + \hat{k}(-9\sin\varphi\cos\varphi\sin^2\theta - 9\sin\varphi\cos\varphi\sin^2\theta) \\ &= (-9\sin^2\varphi\cos\theta, -9\sin^2\varphi\sin\theta, -9\sin\varphi\cos\varphi)\end{aligned}$$

d. Fix the orientation of the normal (if necessary):

$$\vec{N} = (9\sin^2\varphi\cos\theta, \quad 9\sin^2\varphi\sin\theta, \quad 9\sin\varphi\cos\varphi)$$

e. Evaluate the vector field on the cylinder:

$$\sqrt{x^2+y^2} = \sqrt{9\sin^2\varphi\cos^2\theta + 9\sin^2\varphi\sin^2\theta} = 3\sin\varphi$$

$$\vec{F}\left(\vec{R}(\theta,\varphi)\right) = \left(\frac{3\sin\varphi\cos\theta}{3\sin\varphi}, \quad \frac{3\sin\varphi\sin\theta}{3\sin\varphi}, \quad \frac{3\cos\varphi}{3\sin\varphi} \right) = \left(\cos\theta, \quad \sin\theta, \quad \frac{\cos\varphi}{\sin\varphi} \right)$$

f. Evaluate the dot product:

$$\vec{F} \cdot \vec{N} = 9\sin^2\varphi\cos^2\theta + 9\sin^2\varphi\sin^2\theta + 9\cos^2\varphi = 9\sin^2\varphi + 9\cos^2\varphi = 9$$

g. Calculate the flux:

$$\iint_H \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{N}\, d\theta\, d\varphi = \int_0^{\pi/2} \int_0^{2\pi} 9\, d\theta\, d\varphi = 9 \cdot \frac{\pi}{2} \cdot 2\pi = 9\pi^2$$