MATH 253 Honors EXAM 2 Spring 1999

Sections 201-203 Solutions P. Yasskin

1. Compute: $\int_0^1 \int_{y^3}^{y} e^{x/y}\,dx\,dy.$

a. $\frac{1}{2}e^2 - 1$

b. $e + \frac{1}{2}$

c. $e - \frac{1}{2}$

d. $\frac{1}{2}$ correctchoice

e. $\frac{3}{2}$

$$\int_0^1 \int_{y^3}^{y} e^{x/y}\,dx\,dy = \int_0^1 \Big[y e^{x/y} \Big]_{x=y^3}^{y} dy = \int_0^1 \big(y e - y e^{y^2} \big)\,dy = \left[\frac{1}{2} y^2 e - \frac{1}{2} e^{y^2} \right]_{y=0}^{1}$$
$$= \left[\frac{1}{2} e - \frac{1}{2} e^1 \right] - \left[\frac{1}{2} 0e - \frac{1}{2} e^0 \right] = \frac{1}{2}$$

2. Compute $\iint \sin(x^2 + y^2)\,dA$ over the region inside the circle $x^2 + y^2 = \pi$.

a. $\frac{\sqrt{\pi}}{2}$

b. $\sqrt{\pi}$

c. $\frac{\pi}{2}$

d. π

e. 2π correctchoice

$$\iint \sin(x^2 + y^2)\,dA = \int_0^{2\pi} \int_0^{\sqrt{\pi}} \sin(r^2)\,r\,dr\,d\theta = 2\pi \Big[-\frac{1}{2} \cos(r^2) \Big]_{r=0}^{\sqrt{\pi}}$$
$$= 2\pi \Big[-\frac{1}{2} \cos(\pi) \Big] - 2\pi \Big[-\frac{1}{2} \cos(0) \Big] = \pi[- -1 - -1] = 2\pi$$

3. Compute: $\int_0^1 \int_{\sqrt{y}}^1 \frac{3}{1+x^3}\, dx\, dy.$

a. $\ln 2$ correctchoice

b. $\ln 3$

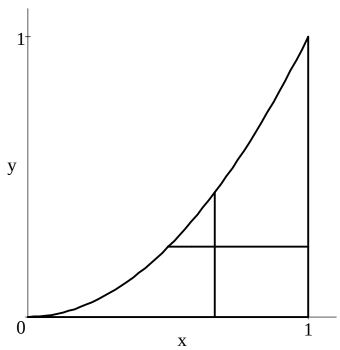
c. $3\ln 2$

d. $2\ln 3$

e. $\frac{3}{2}$

Reverse the order of integration.

$\sqrt{y} \le x \le 1$ becomes $0 \le y \le x^2$



$$\int_0^1 \int_{\sqrt{y}}^1 \frac{3}{1+x^3}\, dx\, dy = \int_0^1 \int_0^{x^2} \frac{3}{1+x^3}\, dy\, dx = \int_0^1 \frac{3y}{1+x^3}\bigg|_{y=0}^{x^2} dx$$

$$= \int_0^1 \frac{3x^2}{1+x^3}\, dx = \ln(1+x^3)\bigg|_{x=0}^{1} = \ln 2 - \ln 1 = \ln 2$$

4. Find the volume between the paraboloids $z = 18 - x^2 - y^2$ and $z = x^2 + y^2$.

a. 36π

b. 81π correctchoice

c. 162π

d. $\frac{243}{2}\pi$

e. 243π

The paraboloids intersect when $18 - x^2 - y^2 = x^2 + y^2$ which is the circle $x^2 + y^2 = 9$.

$$V = \iint (18 - x^2 - y^2) - (x^2 + y^2)\, dA = \int_0^{2\pi} \int_0^3 (18 - 2r^2)\, r\, dr\, d\theta$$

$$= 2\pi \left[9r^2 - \frac{1}{2} r^4 \right]_{r=0}^{3} = 2\pi \left[81 - \frac{81}{2} \right] = 81\pi$$

5. (20 points) Find the point in the **first octant** on the sphere $x^2 + y^2 + z^2 = 9$ at which the function $f = x^4 y^3 z^2$ is a maximum.

$f = x^4 y^3 z^2$ $\qquad \vec{\nabla} f = \left(4x^3y^3z^2,\ 3x^4y^2z^2,\ 2x^4y^3z\right)$

$g = x^2 + y^2 + z^2$ $\qquad \vec{\nabla} g = (2x, \quad 2y, \quad 2z)$

Lagrange equations: $\vec{\nabla} f = \lambda \vec{\nabla} g$:

$4x^3y^3z^2 = 2\lambda x, \qquad 3x^4y^2z^2 = 2\lambda y, \qquad 2x^4y^3z = 2\lambda z$

Solve for λ and eliminate it:

$\lambda = 2x^2y^3z^2 = \frac{3}{2}x^4yz^2 = x^4y^3$

Simplify and solve for x^2 and z^2:

$2y^2 = \frac{3}{2}x^2 \qquad \frac{3}{2}z^2 = y^2 \qquad x^2 = \frac{4}{3}y^2 \qquad z^2 = \frac{2}{3}y^2$

Plug into the constraint and solve for y: (Positive roots for first octant)

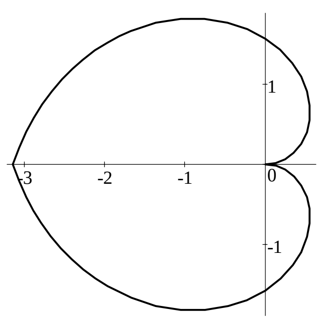
$\frac{4}{3}y^2 + y^2 + \frac{2}{3}y^2 = 9 \qquad \frac{4+3+2}{3}y^2 = 9 \qquad y^2 = 3 \qquad y = \sqrt{3}$

Plug back to find x and z:

$x^2 = \frac{4}{3}y^2 = 4 \qquad x = 2 \qquad z^2 = \frac{2}{3}y^2 = 2 \qquad z = \sqrt{2}$

Solution: $\qquad (x,y,z) = \left(2, \sqrt{3}, \sqrt{2}\right)$

6. (21 points) The heart shape below is the graph of the polar equation $r = |\theta|$ for $-\pi \le \theta \le \pi$. Find the area and the centroid. (16 points for formulas)



$$A = \iint 1\, dA = \int_{-\pi}^{\pi} \int_0^{|\theta|} r\, dr\, d\theta = \int_{-\pi}^{\pi} \left[\frac{1}{2}r^2\right]_{r=0}^{|\theta|} d\theta = \int_{-\pi}^{\pi} \frac{1}{2}\theta^2\, d\theta = \left[\frac{1}{6}\theta^3\right]_{\theta=-\pi}^{\pi} = \frac{1}{3}\pi^3$$

$\bar{y} = 0$ by symmetry.

$$x\text{-mom} = \iint x\, dA = \int_{-\pi}^{\pi} \int_0^{|\theta|} r^2 \cos\theta\, dr\, d\theta = \int_{-\pi}^{\pi} \left[\frac{r^3}{3}\right]_{r=0}^{|\theta|} \cos\theta\, d\theta$$

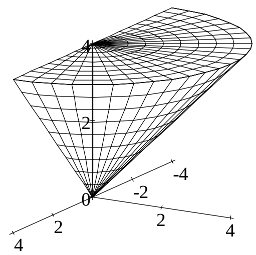
$$= \int_{-\pi}^{\pi} \frac{|\theta|^3}{3} \cos\theta\, d\theta = 2\int_0^{\pi} \frac{\theta^3}{3} \cos\theta\, d\theta$$

Integrate by parts 3 times to get

$$x\text{-mom} = \left[\frac{2}{3}\theta^3 \sin\theta + 2\theta^2\cos\theta - 4\theta\sin\theta - 4\cos\theta\right]_{\theta=0}^{\pi} = 8 - 2\pi^2$$

$$\bar{x} = \frac{x\text{-mom}}{A} = \frac{8-2\pi^2}{\frac{1}{3}\pi^3} = \frac{24-6\pi^2}{\pi^3} \approx -1.136$$

7. (23 points) Find the volume and the centroid of the half of the cone $\sqrt{x^2+y^2} \le z \le 4$ for $y \ge 0$. (15 points for formulas)



$$V = \iiint 1\,dV = \int_0^\pi \int_0^4 \int_r^4 r\,dz\,dr\,d\theta = \pi \int_0^4 \Big[rz \Big]_{z=r}^{4} dr == \pi \int_0^4 4r - r^2\,dr$$

$$= \pi\left[2r^2 - \frac{r^3}{3} \right]_{r=0}^{4} = \pi\left[32 - \frac{64}{3} \right] = \frac{32\pi}{3}$$

$\bar{x} = 0$ by symmetry

$$y\text{-mom} = \iiint y\,dV = \int_0^\pi \int_0^4 \int_r^4 r^2 \sin\theta\,dz\,dr\,d\theta = [-\cos\theta]_{\theta=0}^{\pi} \int_0^4 \Big[r^2 z \Big]_{z=r}^{4} dr$$

$$= 2\int_0^4 (4r^2 - r^3)dr = 2\left[\frac{4r^3}{3} - \frac{r^4}{4} \right]_{r=0}^{4} = 2\left[\frac{256}{3} - \frac{256}{4} \right] = \frac{128}{3}$$

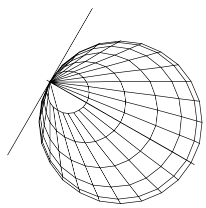
$$\bar{y} = \frac{y\text{-mom}}{A} = \frac{128}{3}\,\frac{3}{32\pi} = \frac{4}{\pi}$$

$$z\text{-mom} = \iiint z\,dV = \int_0^\pi \int_0^4 \int_r^4 rz\,dz\,dr\,d\theta = \pi \int_0^4 \left[r\frac{z^2}{2} \right]_{z=r}^{4} dr == \pi \int_0^4 8r - \frac{1}{2}r^3\,dr$$

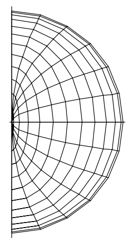
$$= \pi\left[4r^2 - \frac{r^4}{8} \right]_{r=0}^{4} = \pi[64 - 32] = 32\pi$$

$$\bar{z} = \frac{z\text{-mom}}{A} = \frac{32\pi}{1}\,\frac{3}{32\pi} = 3$$

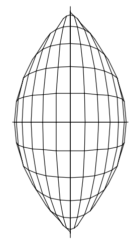
8. (22 points) The graph of the spherical equation $\rho = \sin\theta$ for $0 \le \theta \le \pi$ is shown from the positive z-axis, the positive x-axis, the positive y-axis and in perspective. Find the volume and centroid. (17 points for formulas)



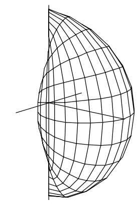
z-axis



x-axis



y-axis



perspective

$$V = \iiint 1\,dV = \int_0^\pi \int_0^\pi \int_0^{\sin\theta} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_0^\pi \int_0^\pi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\sin\theta} \sin\phi \, d\theta \, d\phi$$

$$= \int_0^\pi \int_0^\pi \frac{\sin^3\theta}{3} \sin\phi \, d\theta \, d\phi = \frac{1}{3}\Big[-\cos\phi \Big]_{\phi=0}^{\pi} \left[-\cos\theta + \frac{\cos^3\theta}{3} \right]_{\theta=0}^{\pi}$$

$$= \frac{1}{3}[2]\left[\frac{4}{3} \right] = \frac{8}{9}$$

$\bar{x} = \bar{z} = 0$ by symmetry

$$y\text{-mom} = \iiint y\,dV = \int_0^\pi \int_0^\pi \int_0^{\sin\theta} \rho^3 \sin^2\phi \sin\theta \, d\rho \, d\theta \, d\phi = \int_0^\pi \int_0^\pi \left[\frac{\rho^4}{4} \right]_{\rho=0}^{\sin\theta} \sin^2\phi \sin\theta \, d\theta \, d\phi$$

$$= \frac{1}{4} \int_0^\pi \sin^2\phi \, d\phi \int_0^\pi \sin^5\theta \, d\theta = \frac{1}{4}\frac{1}{2}[\pi] \int_0^\pi (1 - \cos^2\theta)^2 \sin\theta \, d\theta \qquad u = \cos\theta$$

$$= -\frac{\pi}{8} \int_1^{-1} (1 - u^2)^2 \, du = \frac{\pi}{8} \int_{-1}^{1} (1 - 2u^2 + u^4) \, du = \frac{\pi}{8}\left[u - \frac{2u^3}{3} + \frac{u^5}{5} \right]_{u=-1}^{1}$$

$$= \frac{\pi}{4}\left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{\pi}{4}\left(\frac{8}{15} \right) = \frac{2\pi}{15}$$

$$\bar{y} = \frac{y\text{-mom}}{V} = \frac{2\pi}{15}\frac{9}{8} = \frac{3\pi}{20} \approx .471$$