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MATH 308 Exam 1 Spring 2008

Section 511 Hand Computations P. Yasskin

1	/15	6	/10
2	/15	7	/15
3	/15	8	/ 5
4	/ 5	Maple 1	/15
5	/ 5	Maple 2	/15
Total			/100

Problems 1-3: **Do 2 of the 3 problems**, **only**.
Identify the differential equation as one of the following types:

- **a**. Separable Equation
- **b**. Equation with Homogeneous Coefficients
- **c**. Linear Equation
- **d**. Bernoulli Equation
- **e**. Exact Equation

Then solve the initial value problem.

1. (15 points) $\frac{dy}{dx} + \frac{y}{x} = 2x^2y^2$ with $y(1) = \frac{1}{2}$

Bernoulli Equation. Standard form is: $\frac{dy}{dx} + P(x)y = Q(x)y^n$

Here, $n = 2$ and we set $v = y^{1-n} = y^{-1}$. So $y = v^{-1}$ and $\frac{dy}{dx} = -v^{-2}\frac{dv}{dx}$.

The equation becomes: $-v^{-2}\frac{dv}{dx} + \frac{1}{x}v^{-1} = 2x^2v^{-2}$ or $\frac{dv}{dx} - \frac{1}{x}v = -2x^2$

This is linear. $P = \frac{-1}{x}$ $I = e^{\int P dx} = e^{-\ln x} = \frac{1}{x}$

We multiply by the integrating factor: $\frac{1}{x}\frac{dv}{dx} - \frac{1}{x^2}v = -2x$ or $\frac{d}{dx}\left(\frac{v}{x}\right) = -2x$

We integrate: $\frac{v}{x} = -x^2 + C$ $v = -x^3 + Cx$ $y = \frac{1}{v} = \frac{1}{-x^3 + Cx}$

Apply the initial condition: $x = 1$, $y = \frac{1}{2}$, $v = 2 = -x^3 + Cx = -1 + C$

$C = 3$ $y = \frac{1}{-x^3 + 3x}$

2. (15 points) $(-e^{-x} + e^y)\,dx + (xe^y + e^y)\,dy = 0$ with $y(1) = 0$

Exact Equation. Check it is exact: $\frac{d}{dy}(-e^{-x} + e^y) = e^y$ and $\frac{d}{dx}(xe^y + e^y) = e^y$

Since they are equal, it is exact. Find the scalar potential:

$\frac{\partial F}{\partial x} = -e^{-x} + e^y \quad\Rightarrow\quad F = e^{-x} + xe^y + f(y)$

$\frac{\partial F}{\partial y} = xe^y + e^y \quad\Rightarrow\quad F = xe^y + e^y + g(x)$ So $F = xe^y + e^y + e^{-x} = C$

Apply the initial conditions: $x = 1$, $y = 0$ $\Rightarrow$

$F = xe^y + e^y + e^{-x} = 1 + 1 + e^{-1} = C = 2 + e^{-1}$

So the implicit solution is: $xe^y + e^y + e^{-x} = 2 + e^{-1}$

Solve for y: $(x+1)e^y = 2 + e^{-1} - e^{-x}$

$y = \ln\left(\frac{2 + e^{-1} - e^{-x}}{x+1}\right)$

3. (15 points) $\dfrac{dy}{dx} = \dfrac{x^2}{y^2} + \dfrac{y}{x}$ with $y(1) = 3$

Equation with Homogeneous Coefficients.

We set $v = \dfrac{y}{x}$ or $y = xv$ and $\dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$

The equation becomes: $x\dfrac{dv}{dx} + v = \dfrac{1}{v^2} + v$ or $x\dfrac{dv}{dx} = \dfrac{1}{v^2}$

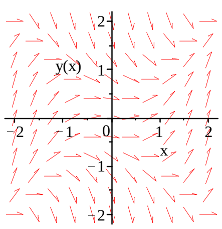
This is separable. We separate: $\int v^2\,dv = \int \dfrac{dx}{x}$ $\quad \dfrac{v^3}{3} = \ln|x| + C$

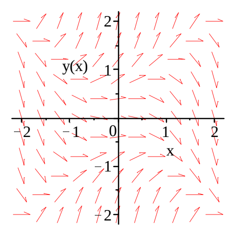
Apply the initial conditions: $x = 1, \quad y = 3, \quad v = \dfrac{y}{x} = 3$

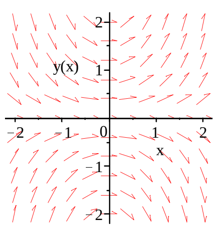
$\dfrac{v^3}{3} = 9 = \ln|x| + C = \ln 1 + C = C \qquad \dfrac{v^3}{3} = \ln|x| + 9$

$v = (3\ln|x| + 27)^{1/3} \qquad y = xv = x(3\ln|x| + 27)^{1/3}$

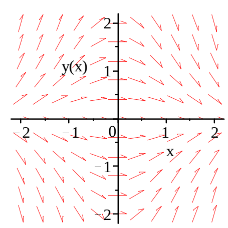
4. (5 points) Which of the following is the direction field of the differential equation:
$\dfrac{dy}{dx} = xy.$ Circle the correct answer:

a. 

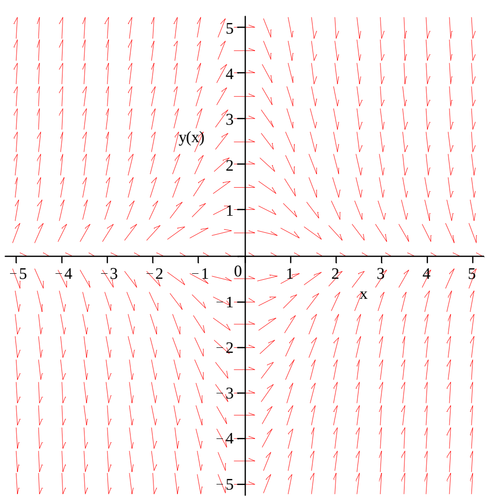
b. 

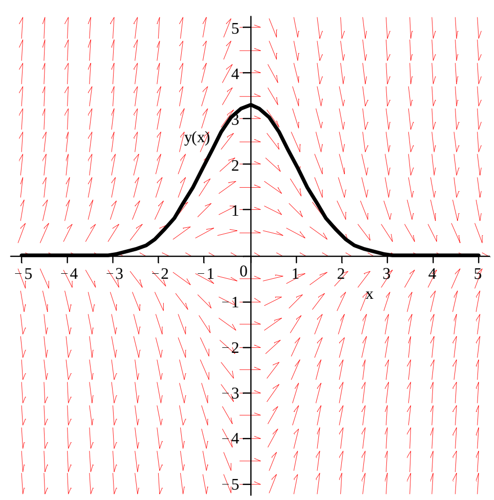
c. 

Correct because it is the only plot with all positive slopes in the first quadrant.

d. 

5. (5 points) On the following direction field, draw the solution curve with the initial condition $y(1) = 2$.





6. (10 points) Find the general solution of the differential equation $x^2 \frac{d^2y}{dx^2} + 6x\frac{dy}{dx} + 6y = 0$

Try $y = x^r$: $x^2 r(r-1)x^{r-2} + 6xrx^{r-1} + 6x^r = 0$ $r(r-1) + 6r + 6 = 0$
$r^2 + 5r + 6 = 0$ $(r+3)(r+2) = 0$ $r = -2, -3$
$y = c_1 x^{-2} + c_2 x^{-3}$

7. (15 points) Consider the initial value problem

$$2\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 26y = 0 \quad \text{with} \quad y(0) = 2 \quad \text{and} \quad \frac{dy}{dt}(0) = -1$$

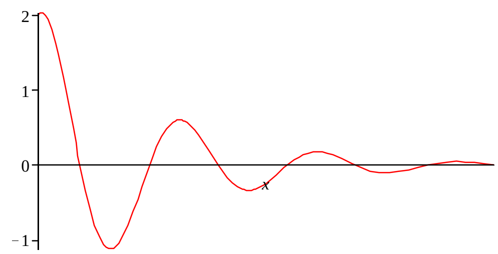
a. (8 pts) Find the general solution of the differential equation.

Try $y = e^{rt}$: $2r^2 + 8r + 26 = 0$ $r = \frac{-8 \pm \sqrt{64 - 208}}{4} = \frac{-8 \pm 12i}{4} = -2 \pm 3i$
General solution: $y = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t)$

b. (7 pts) Find the solution satisfying the initial conditions.

$$y = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t)$$
$$\frac{dy}{dt} = c_1[-2e^{-2t}\cos(3t) - 3e^{-2t}\sin(3t)] + c_2[-2e^{-2t}\sin(3t) + 3e^{-2t}\cos(3t)]$$
$$y(0) = c_1 = 2$$
$$\frac{dy}{dt}(0) = c_1[-2] + c_2[3] = -1 \qquad 3c_2 = -1 + 2c_1 == 3 \qquad c_2 = 1$$
$$y = 2e^{-2t}\cos(3t) + e^{-2t}\sin(3t)$$

8. (5 points) The solution of an initial value problem of the form
$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0 \quad \text{with} \quad y(0) = 2 \quad \text{and} \quad \frac{dy}{dt}(0) = 1$$
is graphed below.



What can you say about the signs and relative sizes of the coefficients, b and k?

Since it oscillates, there are sines and cosines. Since it is damped, there are exponentials with negative powers.

$$y = c_1 e^{\alpha t}\sin(\beta t) + c_2 e^{\alpha t}\cos(\beta t) \qquad \text{with } \alpha < 0$$

The complex roots are $r = \alpha \pm i\beta = \frac{-b \pm \sqrt{b^2 - 4k}}{2}$

To have $\alpha < 0$, we need $b > 0$.

To have complex roots, we need $4k > b^2$.