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MATH 308 Exam 2 Spring 2009

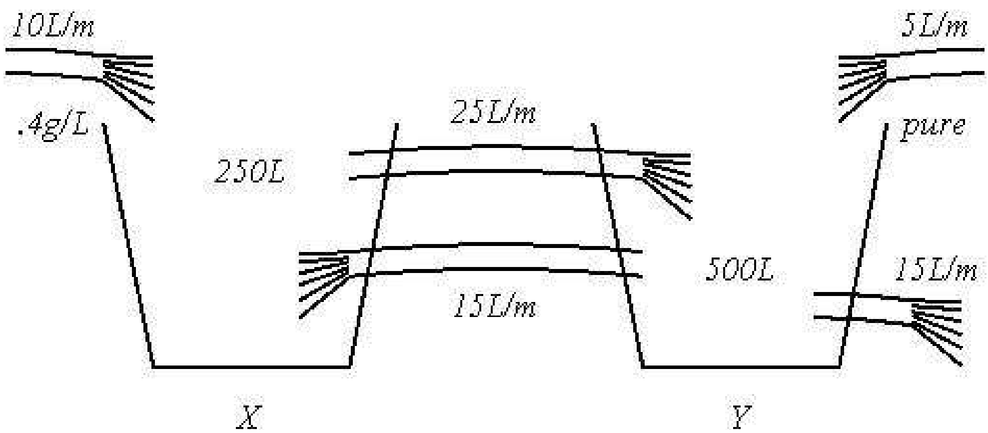
Section 511 Hand Computations P. Yasskin

Solutions

1	/10	4	/30
2	/10	5	/30
3	/10	6	/15
Total			/105

1. (10 points) Tank X initially contains 250 L of sugar water with concentration 0.3 gm/L.
Tank Y initially contains 500 L of sugar water with concentration 0.2 gm/L.
Sugar water with concentration 0.4 gm/L is flowing into tank X at 10 L/min.
Pure water is flowing into tank Y at 5 L/min.
Sugar water is pumped from tank X to tank Y at 25 L/min.
Sugar water is pumped from tank Y to tank X at 15 L/min.
Finally, sugar water is draining from tank Y at 15 L/min.

Draw a **figure**. Define your **variables**.
Set up the **differential equations** and **initial conditions**.
Do not solve the equations.



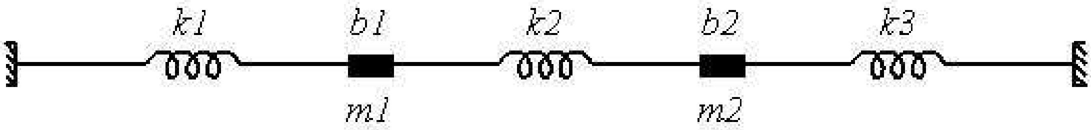
Let $x(t)$ be the gm's of sugar in tank X. Let $y(t)$ be the gm's of sugar in tank Y.

$$\frac{dx}{dt} = \underbrace{\left(0.4\frac{\text{gm}}{\text{L}}\,10\frac{\text{L}}{\text{min}} + \frac{y\text{ gm}}{500\text{ L}}\,15\frac{\text{L}}{\text{min}}\right)}_{\text{in}} - \underbrace{\left(\frac{x\text{ gm}}{250\text{ L}}\,25\frac{\text{L}}{\text{min}}\right)}_{\text{out}} \qquad \boxed{\frac{dx}{dt} = 4 + \frac{3}{100}y - \frac{1}{10}x}$$

$$\frac{dy}{dt} = \underbrace{\left(\frac{x\text{ gm}}{250\text{ L}}\,25\frac{\text{L}}{\text{min}}\right)}_{\text{in}} - \underbrace{\left(\frac{y\text{ gm}}{500\text{ L}}\,(15+15)\frac{\text{L}}{\text{min}}\right)}_{\text{out}} \qquad \boxed{\frac{dy}{dt} = \frac{1}{10}x - \frac{3}{50}y}$$

$$x(0) = 0.3\frac{\text{gm}}{\text{L}}\,250\text{ L} = 75\text{ gm} \qquad y(0) = 0.2\frac{\text{gm}}{\text{L}}\,500\text{ L} = 100\text{ gm} \qquad \boxed{x(0) = 75 \quad y(0) = 100}$$

2. (10 points) Consider the mass and spring system shown in the figure.



The masses are $m_1 = 2$ kg and $m_2 = 3$ kg.
The spring constants are $k_1 = 4$ N/m, $k_2 = 5$ N/m and $k_3 = 6$ N/m.
The drag coefficients are $b_1 = 7$ N•sec/m and $b_2 = 8$ N•sec/m.
Initially, mass m_1 is moved 2 m to the left and given a velocity of 4 m/sec to the right, while mass m_2 is moved 3 m to the right and given a velocity of 5 m/sec to the left.

Define your **variables**. Set up the **differential equations** and **initial conditions**.
Do not solve the equations.

Let $x(t)$ be the displacement of m_1 from it's rest position measured positive to the right.
Let $y(t)$ be the displacement of m_2 from it's rest position measured positive to the right.

The motion of m_1 is given by:

$$m_1 x'' = -k_1 x - b_1 x' + k_2(y - x) \qquad \text{or} \qquad \boxed{2x'' = -4x - 7x' + 5(y - x)}$$

Here, the coefficient of k_1 is negative because when m_1 is to the right of its rest position, $(x > 0)$ the spring is stretched and pulls to the left. Similarly, the coefficient of b_1 is negative because when m_1 is moving to the right, $(x' > 0)$ the force is to the left. And finally, the coefficient of k_2 is positive because when m_2 is more to the right than m_1, $\left(y > x \text{ or } y - x > 0\right)$
the spring is stretched and pulls inward so that m_1 is pulled to the right.

The motion of m_2 is given by:

$$m_2 y'' = -k_3 y - b_2 y' - k_2(y - x) \qquad \text{or} \qquad \boxed{3y'' = -6y - 8y' - 5(y - x)}$$

Here, the coefficient of k_3 is negative because when m_2 is to the right of its rest position, $(y > 0)$ the spring is compressed and pushes to the left. The coefficient of b_2 is negative because when m_2 is moving to the right, $(y' > 0)$ the force is to the left. And finally, the coefficient of k_2 is positive because when m_2 is more to the right than m_1, $\left(y > x \text{ or } y - x > 0\right)$
the spring is stretched and pulls inward so that m_2 is pulled to the left.

The initial conditions are:

$$\boxed{x(0) = -2 \qquad x'(0) = 4 \qquad y(0) = 3 \qquad y'(0) = -5}$$

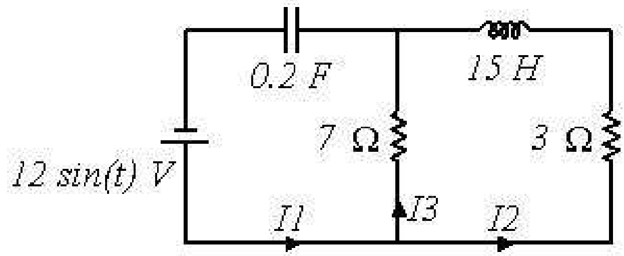
3. (10 points) Consider the circuit shown.

Set up the **equations** for the system.

Do not solve the equations.

Give an **algebraic equation** for I_3 and **differential equations** for I_1 and I_2.

The charge on the capacitor and I_3 should **NOT** appear in the equations for I_1 and I_2.



From the node at the center of the bottom: $I_1 = I_2 + I_3$ So: $\boxed{I_3 = I_1 - I_2}$

From the left circuit: $I_3 R_7 + \dfrac{Q_1}{C} = V$ Or: $7(I_1 - I_2) + 5Q_1 = 12\sin t$

Differentiating and using $\dfrac{dQ_1}{dt} = I_1$ we have: $\boxed{7\left(\dfrac{dI_1}{dt} - \dfrac{dI_2}{dt}\right) + 5I_1 = 12\cos t}$

From the right circuit: $L\dfrac{dI_2}{dt} + I_2 R_3 = I_3 R_7$ Or: $\boxed{15\dfrac{dI_2}{dt} + 3I_2 = 7(I_1 - I_2)}$

To separate the second derivatives, we have:

$15\dfrac{dI_2}{dt} = 7I_1 - 10I_2$ Or: $\boxed{\dfrac{dI_2}{dt} = \dfrac{7}{15}I_1 - \dfrac{2}{3}I_2}$

$\dfrac{dI_1}{dt} = \dfrac{dI_2}{dt} - \dfrac{5}{7}I_1 + \dfrac{12}{7}\cos t$ Or: $\dfrac{dI_1}{dt} = \dfrac{7}{15}I_1 - \dfrac{2}{3}I_2 - \dfrac{5}{7}I_1 + \dfrac{12}{7}\cos t$

Or: $\dfrac{dI_1}{dt} = \dfrac{7}{15}I_1 - \dfrac{2}{3}I_2 - \dfrac{5}{7}I_1 + \dfrac{12}{7}\cos t$

Or: $\boxed{\dfrac{dI_1}{dt} = -\dfrac{26}{105}I_1 - \dfrac{2}{3}I_2 + \dfrac{12}{7}\cos t}$

4. (30 points) Solve the system of differential equations and initial conditions.

$$\frac{dx}{dt} = -4x + 2y + 2 \qquad x(0) = \frac{7}{4}$$

$$\frac{dy}{dt} = 4x - 6y \qquad y(0) = \frac{9}{2}$$

HINTS: Write the equations in operator form.
Cross multiply by operators to separate the variables.
Use the characteristic polynomial to find the homogeneous solution for one variable.
Use undetermined coefficients to find the particular solution for that variable.
Compute the other variable.
Use the initial conditions to determine the constants of integration.

Method 1: Eliminate x:

$$(D+4)x - 2y = 2 \qquad \times 4 \qquad 4(D+4)x - 8y = 8$$

$$-4x + (D+6)y = 0 \qquad \times(D+4) \qquad -4(D+4)x + (D+4)(D+6)y = 0$$

Add: $(D^2 + 10D + 24 - 8)y = 8 \qquad y'' + 10y' + 16y = 8$

Characteristic eq: $r^2 + 10r + 16 = 0 \qquad (r+2)(r+8) = 0 \qquad r = -2, -8 \qquad y_h = c_1 e^{-2t} + c_2 e^{-8t}$

Since the $RHS = 8$ is a constant, we guess $y_p = A$, a constant. Plug in:

$y_p'' + 10y_p' + 16y_p = 8 \qquad 16A = 8 \qquad A = \frac{1}{2} \qquad \boxed{y = c_1 e^{-2t} + c_2 e^{-8t} + \frac{1}{2}} \qquad y' = -2c_1 e^{-2t} - 8c_2 e^{-8t}$

$x = \frac{1}{4}(y' + 6y) = \frac{1}{4}(-2c_1 e^{-2t} - 8c_2 e^{-8t} + 6c_1 e^{-2t} + 6c_2 e^{-8t} + 3) \qquad \boxed{x = c_1 e^{-2t} - \frac{1}{2}c_2 e^{-8t} + \frac{3}{4}}$

$x(0) = c_1 - \frac{1}{2}c_2 + \frac{3}{4} = \frac{7}{4} \qquad c_1 - \frac{1}{2}c_2 = 1$

$y(0) = c_1 + c_2 + \frac{1}{2} = \frac{9}{2} \qquad c_1 + c_2 = 4$

Subtract: $\frac{3}{2}c_2 = 3 \qquad c_2 = 2$

$c_1 = 4 - c_2 = 2$

$$\boxed{x = 2e^{-2t} - e^{-8t} + \frac{3}{4} \qquad y = 2e^{-2t} + 2e^{-8t} + \frac{1}{2}}$$

Method 2: Eliminate y:

$$(D+4)x - 2y = 2 \qquad \times(D+6) \qquad (D+6)(D+4)x - 2(D+6)y = (D+6)2$$

$$-4x + (D+6)y = 0 \qquad \times 2 \qquad -8x + 2(D+6)y = 0$$

Add: $(D^2 + 10D + 24 - 8)x = 12 \qquad x'' + 10x' + 16x = 12$

Characteristic eq: $r^2 + 10r + 16 = 0 \qquad (r+2)(r+8) = 0 \qquad r = -2, -8 \qquad x_h = c_1 e^{-2t} + c_2 e^{-8t}$

Since the $RHS = 12$ is a constant, we guess $x_p = A$, a constant. Plug in:

$x_p'' + 10x_p' + 16x_p = 12 \qquad 16A = 12 \qquad A = \frac{3}{4} \qquad \boxed{x = c_1 e^{-2t} + c_2 e^{-8t} + \frac{3}{4}} \qquad x' = -2c_1 e^{-2t} - 8c_2 e^{-8t}$

$y = \frac{1}{2}x' + 2x - 1 = \frac{1}{2}(-2c_1 e^{-2t} - 8c_2 e^{-8t}) + 2\left(c_1 e^{-2t} + c_2 e^{-8t} + \frac{3}{4}\right) - 1 \qquad \boxed{y = c_1 e^{-2t} - 2c_2 e^{-8t} + \frac{1}{2}}$

$x(0) = c_1 + c_2 + \frac{3}{4} = \frac{7}{4} \qquad c_1 + c_2 = 1$

$y(0) = c_1 - 2c_2 + \frac{1}{2} = \frac{9}{2} \qquad c_1 - 2c_2 = 4$

Subtract: $3c_2 = -3 \qquad c_2 = -1$

$c_1 = 1 - c_2 = 2$

$$\boxed{x = 2e^{-2t} - e^{-8t} + \frac{3}{4} \qquad y = 2e^{-2t} + 2e^{-8t} + \frac{1}{2}}$$

5. (30 points) Let $L(y) = y'' - (\tan x)y' - (\sec^2 x)y$. Find the general solution of the differential equation $L(y) = 3\sec^3 x \tan x$ by completing the following steps:

a. Show $y_1 = \sec x$ and $y_2 = \tan x$ are solutions of the homogeneous equation $L(y) = 0$.

$y_1' = \sec x \tan x \qquad y_1'' = \sec x \tan^2 x + \sec^3 x \qquad L(y_1) = \sec x \tan^2 x + \sec^3 x - \sec x \tan^2 x - \sec^3 x = 0$

$y_2' = \sec^2 x \qquad y_2'' = 2\sec^2 x \tan x \qquad L(y_2) = 2\sec^2 x \tan x - \sec^2 x \tan x - \sec^2 x \tan x = 0$

b. Show $y_1 = \sec x$ and $y_2 = \tan x$ are linearly independent.

Method 1: Assume $a\sec x + b\tan x = 0$ for all x.

Then: At $x = 0$: $a = 0$. At $x = \frac{\pi}{4}$: $a\sqrt{2} + b = 0$

So $a = b = 0$. So $\sec x$ and $\tan x$ are linearly independent.

Method 2: Compute the Wronskian:

$$W(x) = \begin{vmatrix} \sec x & \tan x \\ \sec x \tan x & \sec^2 x \end{vmatrix} = \sec^3 x - \sec x \tan^2 x = \sec x(\sec^2 x - \tan^2 x) = \sec x \neq 0 \quad \text{for all } x.$$

Method 3: Compute the Wronskian at one point:

$$W(x) = \begin{vmatrix} \sec x & \tan x \\ \sec x \tan x & \sec^2 x \end{vmatrix} \qquad W(0) = \begin{vmatrix} \sec 0 & \tan 0 \\ \sec 0 \tan 0 & \sec^2 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

c. Write out the general homogeneous solution.

$\boxed{y_h = c_1 \sec x + c_2 \tan x}$

d. Use variation of parameters to find a particular solution.

$y_p = v_1(x)\sec x + v_2(x)\tan x$

We must solve the equations:

$$v_1'(x)\sec x + v_2'(x)\tan x = 0 \qquad (1)$$

$$v_1'(x)\sec x \tan x + v_2'(x)\sec^2 x = 3\sec^3 x \tan x \qquad (2)$$

Multiply (1) by $-\tan x$ and add (2): $v_2'(x)(\sec^2 x - \tan^2 x) = 3\sec^3 x \tan x$

But $\sec^2 x - \tan^2 x = 1$. So $v_2'(x) = 3\sec^3 x \tan x$ or $\boxed{v_2(x) = \int 3\sec^3 x \tan x \, dx}$.

Let $u = \sec x$, $du = \sec x \tan x \, dx$. Then $v_2(x) = \int 3u^2 \, du = u^3 = \sec^3 x$.

From the first equation: $v_1'(x) = -\frac{\tan x}{\sec x} v_2'(x) = -\frac{\tan x}{\sec x} 3\sec^3 x \tan x = -3\sec^2 x \tan^2 x$

So $\boxed{v_1(x) = -\int 3\sec^2 x \tan^2 x \, dx}$.

Let $u = \tan x$, $du = \sec^2 x \, dx$. Then $v_1(x) = -\int 3u^2 \, du = -u^3 = -\tan^3 x$.

$y_p = -\tan^3 x \sec x + \sec^3 x \tan x = \sec x \tan x(\sec^2 x - \tan^2 x) \qquad \boxed{y_p = \sec x \tan x}$

e. Write out the general non-homogeneous solution.

$\boxed{y = c_1 \sec x + c_2 \tan x + \sec x \tan x}$

6. (15 points) Find the solution of the initial value problem:

$$\frac{d^3y}{dt^3} + 4\frac{dy}{dt} = 4\cos(2t) \quad \text{with} \quad y(0) = 2 \quad \text{and} \quad \frac{dy}{dt}(0) = \frac{1}{2} \quad \text{and} \quad \frac{d^2y}{dt^2}(0) = 4$$

The characteristic equation for the homogeneous solution is

$r^3 + 4r = 0$ Or: $r(r^2 + 4) = r(r - 2i)(r + 2i) = 0$

The roots are: $r = 0, 2i, -2i$

So the homogeneous solution is $\boxed{y_h = c_1 + c_2\cos(2t) + c_3\sin(2t)}$

Since the RHS is $3\cos(2t)$, we would normally guess that the particular solution has the form "$y_p = A\cos(2t) + B\sin(2t)$".

However since $\cos(2t)$ is a fundamental solution, we guess that the particular solution has the form $\boxed{y_p = At\cos(2t) + Bt\sin(2t)}$.

To find A and B, we compute the derivatives:

$$y_p' = A\cos(2t) - 2At\sin(2t) + B\sin(2t) + 2Bt\cos(2t)$$

$$\begin{aligned} y_p'' &= -2A\sin(2t) - 2A\sin(2t) - 4At\cos(2t) + 2B\cos(2t) + 2B\cos(2t) - 4Bt\sin(2t) \\ &= -4A\sin(2t) - 4At\cos(2t) + 4B\cos(2t) - 4Bt\sin(2t) \end{aligned}$$

$$\begin{aligned} y_p''' &= -8A\cos(2t) - 4A\cos(2t) + 8At\sin(2t) - 8B\sin(2t) - 4B\sin(2t) - 8Bt\cos(2t) \\ &= -12A\cos(2t) + 8At\sin(2t) - 12B\sin(2t) - 8Bt\cos(2t) \end{aligned}$$

and substitute into the equation:

$$\begin{aligned} &[-12A\cos(2t) + 8At\sin(2t) - 12B\sin(2t) - 8Bt\cos(2t)] \\ &\quad + 4[A\cos(2t) - 2At\sin(2t) + B\sin(2t) + 2Bt\cos(2t)] = 4\cos(2t) \end{aligned}$$

We equate coefficients of the 4 linearly independent functions:

Coeff of $\cos(2t)$: $-12A + 4A = 4 \quad \Rightarrow \quad A = -\frac{1}{2}$

Coeff of $t\cos(2t)$: $-8B + 8B = 0 \quad \Rightarrow$ No information

Coeff of $\sin(2t)$: $-12B + 4B = 0 \quad \Rightarrow \quad B = 0$

Coeff of $t\sin(2t)$: $8A - 8A = 0 \quad \Rightarrow$ No information

So the particular solution is: $\boxed{y_p = -\frac{1}{2}t\cos(2t)}$

The general non-homogeneous solution is

$$\boxed{y = y_h + y_p = c_1 + c_2\cos(2t) + c_3\sin(2t) - \frac{1}{2}t\cos(2t)}$$

To find c_1, c_2 and c_3, we compute the derivatives:

$$y' = -2c_2\sin(2t) + 2c_3\cos(2t) - \frac{1}{2}\cos(2t) + t\sin(2t)$$

$$\begin{aligned} y'' &= -4c_2\cos(2t) - 4c_3\sin(2t) + \sin(2t) + \sin(2t) + 2t\cos(2t) \\ &= -4c_2\cos(2t) - 4c_3\sin(2t) + 2\sin(2t) + 2t\cos(2t) \end{aligned}$$

and use the initial conditions:

$$\begin{aligned} y(0) &= c_1 + c_2 = 2 \\ y'(0) &= 2c_3 - \frac{1}{2} = \frac{1}{2} \\ y''(0) &= -4c_2 = 4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_2 &= -1 \\ c_1 &= 3 \\ c_3 &= \frac{1}{2} \end{aligned}$$

So the solution is: $\boxed{y = 3 - \cos(2t) + \frac{1}{2}\sin(2t) - \frac{1}{2}t\cos(2t)}$