Name________________________ NetID__________________

MATH 308 Final Exam Spring 2009

Section 511 P. Yasskin

1	/20	4	/10
2	/20	5	/20
3	/10	6	/20
Total			/100

1. (20 points) Consider the second order non-homogeneous differential equation

$$y'' + 5y' + 4y = 6e^{-x} + 8x^2 + 3.$$

a. Find two solutions of the homogeneous differential equation $y'' + 5y' + 4y = 0$. Verify they are linearly independent. Give the general homogeneous solution.

b. Use undetermined coefficients to find a particular non-homogeneous solution.

c. Find the non-homogeneous solution satisfying the initial conditions: $y(0) = 6 \quad y'(0) = 3$.

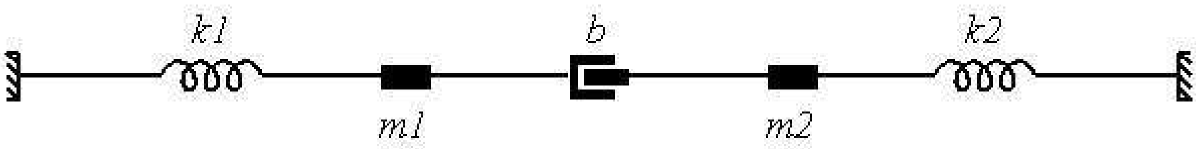
2. (20 points) Consider the second order homogeneous differential equation

$$x^2y'' + (-2x - 2x^2)y' + (2 + 2x + x^2)y = 0.$$

a. Verify $y_1 = xe^x$ is a solution. (Show your algebra!)

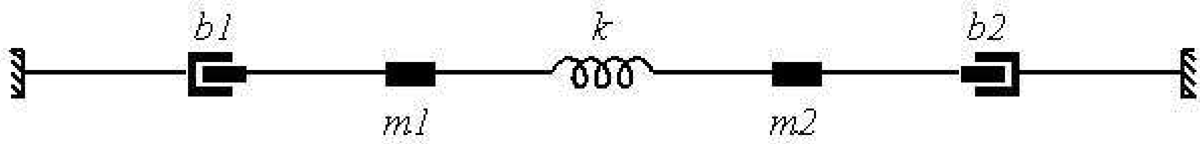
b. Use reduction of order (similar to variation of parameters) to find a second solution. (Be careful with your algebra. Nearly everything should cancel.)

3. (10 points) Consider the mass-spring-piston system shown in the figure.



The masses are $m_1 = 6$ kg and $m_2 = 8$ kg.
The spring constants are $k_1 = 3$ N/m and $k_2 = 5$ N/m.
The piston pulls in with a force proportional to the velocity with which it is expanding
or pushes out with a force proportional to the velocity with which it is contracting
with a proportionality constant which is the drag coefficient of $b = 4$ N•sec/m.
Initially, mass m_1 is moved 2 m to the right and given a velocity of 5 m/sec to the left,
while mass m_2 is moved 4 m to the left and given a velocity of 3 m/sec to the right.
Let $x(t)$ be the displacement of m_1 from it's rest position measured positive to the right.
Let $y(t)$ be the displacement of m_2 from it's rest position measured positive to the right.
Set up **second** order **differential equations** and **initial conditions** for x and y.
Do not solve the equations.

4. (10 points) Consider the mass-spring-piston system shown in the figure.



The masses are $m_1 = 1$ kg and $m_2 = 1$ kg. The spring constants is $k = 7$ N/m.
The drag coefficient for the pistons are $b_1 = 5$ N•sec/m and $b_2 = 6$ N•sec/m.
Each piston pulls in with a force proportional to the velocity with which it is expanding or pushes out with a force proportional to the velocity with which it is contracting.
Initially, mass m_1 is moved 1 m to the right and given a velocity of 2 m/sec to the right, while mass m_2 is moved 3 m to the left and given a velocity of 4 m/sec to the left.
Let $x(t)$ be the displacement of m_1 from it's rest position measured positive to the right.
Let $y(t)$ be the displacement of m_2 from it's rest position measured positive to the right.
The second order differential equations and initial conditions for x and y are

$$m_1 x'' = -b_1 x' + k(y-x) \quad \text{or} \quad \boxed{x'' = -5x' + 7(y-x)} \qquad \boxed{x(0) = 1 \qquad x'(0) = 2}$$

$$m_2 y'' = -b_2 y' - k(y-x) \quad \text{or} \quad \boxed{y'' = -6y' - 7(y-x)} \qquad \boxed{y(0) = -3 \qquad y'(0) = -4}$$

Let $p = x'$ and $q = y'$.
Set up **first** order **differential equations** $\vec{x}' = A\vec{x}$ for $\vec{x} = \begin{pmatrix} x \\ p \\ y \\ q \end{pmatrix}$.
and **initial conditions** for $\vec{x}(0)$.
Do not solve the equations.

5. (20 points) Consider the first order differential equations $\vec{x}' = A\vec{x}$

where $\vec{x} = \begin{pmatrix} x \\ p \\ y \\ q \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -3 \end{pmatrix}$.

Find the **eigenvalues** and **eigenvectors** of A.

Then find the **general solution** of the differential equation. (Vector form is OK.)

HINT: Two of the eigenvalues and eigenvectors are:

$r = -2 \quad \vec{u}_{-2} = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \end{pmatrix} \qquad r = -3 \quad \vec{u}_{-3} = \begin{pmatrix} -1 \\ 3 \\ -1 \\ 3 \end{pmatrix}$

6. (20 points) Consider the first order **non-homogeneous** system of differential equations

$$\vec{x}' = A\vec{x} + \vec{f} \quad \text{where} \quad \vec{x} = \begin{pmatrix} x \\ p \\ y \\ q \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & 3 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \vec{f} = \begin{pmatrix} 3 \\ -t \\ 0 \\ t \end{pmatrix}.$$

The general solution of the corresponding homogeneous differential equation is

$\vec{x} = c_0\vec{u}_0 + c_1 e^{-t}\vec{u}_{-1} + c_2 e^{-2t}\vec{u}_{-2} + c_3 e^{3t}\vec{u}_3$ where

$$\vec{u}_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u_{-1} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad \vec{u}_{-2} = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix}$$

Use undetermined coefficients to determine a **particular solution**.

HINTS: Look for a solution of the form $\vec{x}_p = \vec{a} + t\vec{b}$.

First solve for $\vec{b}$ and keep the arbitrary constant. Then solve for $\vec{a}$.

If you have time check your answer in the differential equation.