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MATH 308	Exam I	Fall 2000
Section 200	Solutions	P. Yasskin

HAND COMPUTATIONS

Problems 1-5: Solve the initial value problem using the method appropriate to one of the following types of first order differential equations:

a. Separable Equation
b. Equation with Homogeneous Coefficients
c. Linear Equation
d. Bernoulli Equation Hint: $k = \dfrac{1}{1-s}$
e. Exact Equation

Be sure to identify the equation type. There is one problem for each equation type.

1. (10 points) $\dfrac{dy}{dx} = \dfrac{4y}{x} + x$ with $y(1) = 0$

Linear Equation. Standard form is:

$$\frac{dy}{dx} - \frac{4}{x}y = x.$$

The integrating factor is

$$I = \exp\left(\int -\frac{4}{x}\,dx\right) = \exp(-4\ln x) = \exp(\ln x^{-4}) = x^{-4}$$

Multiply by the integrating factor and recognize the left as the derivative of a product:

$$\frac{d}{dx}(x^{-4}y) = x^{-4}\frac{dy}{dx} - 4x^{-5}y = x^{-3}$$

Integrate and solve:

$$x^{-4}y = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + C \qquad y = -\frac{x^2}{2} + Cx^4$$

Solve for the constant of integration:

$$x_0 = 1 \qquad y_0 = 0 \qquad \Rightarrow \qquad 0 = -\frac{1}{2} + C \qquad \Rightarrow \qquad C = \frac{1}{2}$$

So the solution is:

$$y = -\frac{x^2}{2} + \frac{1}{2}x^4 = \frac{1}{2}(x^4 - x^2)$$

2. (10 points) $\frac{dy}{dt} = -\frac{2y}{t} + 6t\sqrt{y}$ with $y(1) = 9$

Bernoulli Equation. $s = \frac{1}{2}$, $k = \frac{1}{1-s} = \frac{1}{1-\frac{1}{2}} = 2$, So let $y = z^2$. Then

$$\frac{dy}{dt} = 2z\frac{dz}{dt} = -\frac{2y}{t} + 6t\sqrt{y} = -\frac{2z^2}{t} + 6tz$$

Solve for $\frac{dz}{dt}$ and put the equation into standard linear form:

$$\frac{dz}{dt} = -\frac{z}{t} + 3t \qquad \Rightarrow \qquad \frac{dz}{dt} + \frac{z}{t} = 3t$$

The integrating factor is

$$I = e^{\int \frac{dt}{t}} = e^{\ln t} = t$$

Multiply the equation by the integrating factor and integrate:

$$t\frac{dz}{dt} + z = 3t^2 \qquad \Rightarrow \qquad tz = \int 3t^2 dt = t^3 + C$$

Solve for z and then y:

$$z = t^2 + \frac{C}{t} \qquad \Rightarrow \qquad y = \left(t^2 + \frac{C}{t}\right)^2$$

Use the initial conditions to find C and substitute back

$$x_0 = 1 \qquad y_0 = 9 \qquad \Rightarrow \qquad 9 = (1 + C)^2 \qquad \Rightarrow \qquad C = 2$$

So the solution is:

$$y = \left(t^2 + \frac{2}{t}\right)^2$$

3. (10 points) $\frac{dx}{dt} = \frac{x}{t} + \frac{1}{tx^2}$ with $x(1) = 2$

Separable Equation. Separate the variables:

$$\frac{dx}{dt} = \frac{x^3 + 1}{tx^2} \qquad \Rightarrow \qquad \int \frac{x^2}{x^3 + 1} dx = \int \frac{1}{t} dt \qquad \Rightarrow \qquad \frac{1}{3}\ln|x^3 + 1| = \ln|t| + C$$

Exponentiate:

$$e^{\frac{1}{3}\ln|x^3+1|} = e^{\ln|t|+C} \qquad \Rightarrow \qquad |x^3 + 1|^{1/3} = e^C|t| \qquad \Rightarrow \qquad (x^3 + 1)^{1/3} = At$$

Use the initial conditions to find A:

$$t_0 = 1 \qquad x_0 = 2 \qquad \Rightarrow \qquad (2^3 + 1)^{1/3} = A \qquad \Rightarrow \qquad A = 9^{1/3}$$

Substitute back and solve for x:

$$(x^3 + 1)^{1/3} = 9^{1/3}t \qquad \Rightarrow \qquad x^3 + 1 = 9t^3 \qquad \Rightarrow \qquad x = (9t^3 - 1)^{1/3}$$

4. (10 points) $\dfrac{dy}{dx} = \dfrac{y\sin x - \sin y}{x\cos y + \cos x}$ with $y(0) = 2$

Exact Equation. Rewrite as a differential 1-form:

$$dy = \frac{y\sin x - \sin y}{x\cos y + \cos x}dx \qquad \Rightarrow \qquad -\frac{y\sin x - \sin y}{x\cos y + \cos x}dx + dy = 0$$

Multiply by the integrating factor (which is the denominator):

$$(-y\sin x + \sin y)dx + (x\cos y + \cos x)dy = 0$$

Check it is exact:

$$\frac{d}{dy}(-y\sin x + \sin y) = -\sin x + \cos y \qquad \text{and} \qquad \frac{d}{dx}(x\cos y + \cos x) = \cos y - \sin x$$

They are equal. So it is exact. Find the scalar potential (first integral) G which satisfies

$$dG = (-y\sin x + \sin y)dx + (x\cos y + \cos x)dy$$

or

$$\frac{\partial G}{\partial x} = -y\sin x + \sin y \qquad \text{and} \qquad \frac{\partial G}{\partial y} = x\cos y + \cos x$$

So

$$G = y\cos x + x\sin y$$

and the implicit solution of $dG = 0$ is $G = C$ or

$$y\cos x + x\sin y = C$$

Use the initial conditions to find C:

$$x_0 = 0 \qquad y_0 = 2 \qquad \Rightarrow \qquad 2\cos 0 + 0\sin 2 = C \qquad \Rightarrow \qquad C = 2$$

So the implicit solution is:

$$y\cos x + x\sin y = 2$$

This cannot be solved for y.

5. (10 points) $\dfrac{dy}{dx} = \dfrac{x^2}{y^2} + \dfrac{y}{x}$ with $y(1) = 3$

Equation with Homogeneous Coefficients. Substitute $y(x) = xv(x)$:

$$\frac{dy}{dx} = x\frac{dv}{dx} + v = \frac{x^2}{y^2} + \frac{y}{x} = \frac{1}{v^2} + v \qquad \Rightarrow \qquad x\frac{dv}{dx} = \frac{1}{v^2}$$

Separate variables and integrate:

$$\int v^2\,dv = \int \frac{1}{x}\,dx \qquad \Rightarrow \qquad \frac{v^3}{3} = \ln x + C$$

The initial condition $y(1) = 3$ says $v(1) = 3$. Use this to find C:

$$x_0 = 1 \qquad v_0 = 3 \qquad \Rightarrow \qquad 9 = \ln 1 + C \qquad \Rightarrow \qquad C = 9$$

Substitute back and solve for v and then y:

$$\frac{v^3}{3} = \ln x + 9 \qquad \Rightarrow \qquad v = (3\ln x + 27)^{1/3} \qquad \Rightarrow \qquad y = x(3\ln x + 27)^{1/3}$$

6. (15 points) Set up the differential equation and initial condition for $P(t)$ in the following problem. Do not solve the equations.

A certain pond can contain $12,000$ ft^3 of water before the dam overflows. Initially, there are $8,000$ ft^3 of water and 50 gallons of pollution in the pond. Acme Polluters is putting 2 gallons of pollution in the pond a day. Every day, $2,000$ ft^3 of fresh water is pumped into the pond and $1,000$ ft^3 of polluted water is pumped out. Let $P(t)$ be the gallons of pollution in the pond after t days. When does the dam overflow and what is the concentration of the polution in the water when the dam first overflows?

The ft^3 of water in the pond is

$$W(t) = 8000 + 1000t$$

Differential Equation: (The pure water does not add polution to the pond.)

$$\frac{dP}{dt} = \underbrace{2\frac{\text{gal}}{\text{day}}}_{\text{in}} - \underbrace{\frac{P(t)\text{gal}}{W(t)\text{ft}^3}1000\frac{\text{ft}^3}{\text{day}}}_{\text{out}}$$

or

$$\frac{dP}{dt} = 2 - \frac{1000}{8000 + 1000t}P(t)$$

Initial condition:

$$P(0) = 50$$

MAPLE COMPUTATIONS

7. (20 points) Consider the differential equation $\frac{dy}{dt} = e^t - y$.

a. (4 pts) Find the general solution.

b. (4 pts) Find the specific solution satisfying the initial condition $y(0) = 4$.

c. (4 pts) Plot the direction field of the differential equation for times $-4 \le t \le 6$.

d. (4 pts) Add to the direction field the solutions satisfying each of the initial conditions

$$y(0) = -4, \quad y(0) = -2, \quad y(0) = 0, \quad y(0) = 2, \quad y(0) = 4, \quad y(0) = 6$$

Adjust the vertical range to a reasonable value.

e. (4 pts) Looking at the exact solution and the plot, describe the behavior of the solutions for large times. At about what time does this asymptotic behavior begin?

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8. (15 points) Consider the initial value problem $\frac{dy}{dx} = \cos x + \sin y$ with $y(0) = 1$.

a. (10 pts) What happens if you try to solve the equations using **dsolve** ? Find a Taylor polynomial approximation about $x = 0$ for the solution to this initial value problem keeping terms up to and including x^4.

b. (5 pts) Plot the direction field for this differential equation together with the solution satisfying the initial condition. Plot the Taylor polynomial approximation. Combine the two plots into a single plot. On approximately what interval is the Taylor polynomial a good approximation?