MATH 308 | Exam III | Fall 2000
Section 200 | Solutions | P. Yasskin

HAND COMPUTATIONS

1. (40 points) Use the Laplace Tranform Technique to find the Laplace transform of the solution to the initial value problem

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 9 + 6t \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1.$$

Laplace transform of the differential equation:

$$s^2 Y(s) - sy(0) - y'(0) + 3[sY(s) - y(0)] + 2Y(s) = \frac{9}{s} + \frac{6}{s^2}$$

Substitute in the initial conditions:

$$s^2 Y(s) - s - 1 + 3sY(s) - 3 + 2Y(s) = \frac{9}{s} + \frac{6}{s^2}$$

Solve for $Y(s)$:

$$Y(s)[s^2 + 3s + 2] = s + 4 + \frac{9}{s} + \frac{6}{s^2}$$

$$Y(s) = \frac{s + 4 + \frac{9}{s} + \frac{6}{s^2}}{s^2 + 3s + 2}$$

2. (10 points)

a. Find the Laplace transform of the function: $f(t) = te^{2t}\sin 3t$

$$f_1(t) = \sin 3t \qquad F_1(s) = \frac{3}{s^2 + 9}$$

$$f_2(t) = e^{2t}\sin 3t = e^{2t}f_1(t) \qquad F_2(s) = F_1(s-2) = \frac{3}{(s-2)^2 + 9}$$

$$f(t) = te^{2t}\sin 3t = tf_2(t) \qquad F(s) = -\frac{d}{ds}F_2(s) = -\frac{-3[2(s-2)]}{\left((s-2)^2 + 9\right)^2}$$

$$F(s) = \frac{6(s-2)}{\left((s-2)^2 + 9\right)^2}$$

b. Find the inverse Laplace transform of the function: $G(s) = \dfrac{2e^{-2s}}{(s+1)^3}$

$$G_1(s) = \frac{2}{s^3} \qquad g_1(t) = t^2$$

$$G_2(s) = \frac{2}{(s+1)^3} = G_1(s+1) \qquad g_2(t) = e^{-t}g_1(t) = e^{-t}t^2$$

$$G(s) = \frac{2e^{-2s}}{(s+1)^3} = e^{-2s}G_2(s) \qquad g(t) = g_2(t-2)\Theta(t-2)$$

$$g(t) = e^{-(t-2)}(t-2)^2\Theta(t-2)$$

3. (20 points) Find the solution of the system

$$\frac{dx}{dt} = -8x + 8y \qquad x(0) = 6$$
$$\frac{dy}{dt} = -3x + 2y \qquad y(0) = 4$$

using the Eigenvector Technique.

Coefficient matrix:

$$A = \begin{pmatrix} -8 & 8 \\ -3 & 2 \end{pmatrix}$$

Characteristic polynomial:

$$\det(A - \lambda\mathbf{1}) = \begin{vmatrix} -8-\lambda & 8 \\ -3 & 2-\lambda \end{vmatrix} = (-8-\lambda)(2-\lambda) + 24 = \lambda^2 + 6\lambda + 8$$

Characteristic equation and eigenvalues:

$$\lambda^2 + 6\lambda + 8 = 0 \qquad (\lambda+2)(\lambda+4) = 0 \qquad \lambda = -2, -4$$

Find eigenvectors:

$\lambda = -2$:

$$\begin{pmatrix} -6 & 8 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -3u_1 + 4u_2 = 0 \quad \Rightarrow \quad u_2 = \frac{3}{4}u_1 \quad \Rightarrow \quad \vec{u}_{-2} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$\lambda = -4$:

$$\begin{pmatrix} -4 & 8 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -4u_1 + 8u_2 = 0 \quad \Rightarrow \quad u_2 = \frac{1}{2}u_1 \quad \Rightarrow \quad \vec{u}_{-4} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Vector solution:

$$X = Be^{-2t}\vec{u}_{-2} + Ce^{-4t}\vec{u}_{-4} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = Be^{-2t}\begin{pmatrix} 4 \\ 3 \end{pmatrix} + Ce^{-4t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Scalar solution:

$$x(t) = 4Be^{-2t} + 2Ce^{-4t}$$
$$y(t) = 3Be^{-2t} + Ce^{-4t}$$

Use the initial conditions:

$$\left.\begin{aligned} x(0) &= 4B + 2C = 6 \\ y(0) &= 3B + C \;\;= 4 \end{aligned}\right\} \quad \Rightarrow \quad \begin{aligned} B &= 1 \\ C &= 1 \end{aligned}$$

Solution:

$$x(t) = 4e^{-2t} + 2e^{-4t}$$
$$y(t) = 3e^{-2t} + e^{-4t}$$

MAPLE COMPUTATIONS

4. (30 points) Find the solution of the system

$$\frac{dx}{dt} = -8x + 8y \qquad x(0) = 6$$
$$\frac{dy}{dt} = -3x + 2y \qquad y(0) = 4$$

using the Laplace Tranform Technique. (You may not use the **dsolve** command except to check your answer.)

To Turn in Your Maple Computations:

1. Save your Maple file as lastname_exam3.mws

2. Print your file as follows:

- **a**. Click on **FILE**, **PRINT** and **Printer Command**.
- **b**. Make the command read: **lpr -J** ”**Yasskin Maple Exam 3**”
- **c**. Call Dr. Yasskin over to check your printing.
- **d**. Click on **PRINT**.

3. Mail your file as follows:

- **a**. Start the mail program: **pine**
- **b**. Compose a letter by typing **C**.
- **c**. In the header region, enter:
  - To **yasskin**
  - Attachment **lastname_exam3.mws** (or the *exact* name of your Maple file)
  - Subject **Last Name Exam3**
- **d**. Call Dr. Yasskin over to check your email.
- **e**. Mail the letter by typing ^**X** and **Y**.