MATH 311 Exam 2 Fall 2000
Section 502 Solutions P. Yasskin

1. (20 points) Consider the vector space P_2 of polynomials of degree ≤ 2.
Consider the function of two polynomials

$$\langle p,q\rangle = \int_0^1 p(x)\,q(x)\,dx$$

a. (10 pts) Show $\langle p,q\rangle$ is an inner product.

$$\langle q,p\rangle = \int_0^1 q(x)\,p(x)\,dx = \int_0^1 p(x)\,q(x)\,dx = \langle p,q\rangle$$

$\langle p,p\rangle = \int_0^1 p(x)^2\,dx \ge 0$ and $= 0$ only when $p(x) = 0$ for all x.

$$\langle p,aq+br\rangle = \int_0^1 p(x)\,(aq+br)(x)\,dx = a\int_0^1 p(x)\,q(x)\,dx + b\int_0^1 p(x)\,r(x)\,dx = a\langle p,q\rangle + b\langle p,r\rangle$$

b. (10 pts) Find $\cos\theta$ where θ is the angle between the polynomials

$$p = 5x^2+3 \quad \text{and} \quad q = 3x.$$

$$\langle p,q\rangle = \int_0^1 p(x)\,q(x)\,dx = \int_0^1 (5x^2+3)\,(3x)\,dx = \int_0^1 (15x^3+9x)\,dx = \left[\frac{15x^4}{4} + \frac{9x^2}{2}\right]_0^1$$

$$= \frac{15}{4} + \frac{9}{2} = \frac{33}{4}$$

$$\langle p,p\rangle = \int_0^1 p(x)^2\,dx = \int_0^1 (5x^2+3)^2\,dx = \int_0^1 (25x^4+30x^2+9)\,dx = \left[5x^5+10x^3+9x\right]_0^1 = 24$$

$$\langle q,q\rangle = \int_0^1 q(x)^2\,dx = \int_0^1 (3x)^2\,dx = \left[3x^2\right]_0^1 = 3$$

$$\cos\theta = \frac{\langle p,q\rangle}{|p||q|} = \frac{33}{4\sqrt{24}\,\sqrt{3}} = \frac{11}{8\sqrt{2}}$$

2. (40 points) Consider the vector space P_2 of polynomials of degree ≤ 2.
Consider the bases

$$e_1 = 1 \qquad e_2 = x \qquad e_3 = x^2$$

$$f_1 = 1+x \qquad f_2 = x \qquad f_3 = -x+x^2$$

Consider the function $L : P_2 \to P_2$ given by

$$L(p) = 2p(0) + p(1)x$$

a. (5 pts) Show L is linear.

$$L(ap+bq) = 2(ap+bq)(0) + (ap+bq)(1)x = a(2p(0)+p(1)x) + b(2q(0)+q(1)x)$$

$$= aL(p) + bL(q)$$

b. (5 pts) Find the matrix of L relative to the e-basis. Call it $\underset{e\leftarrow e}{A}$.

$$\begin{aligned} L(e_1) &= L(1) &&= 2+x &&= 2e_1+e_2 \\ L(e_2) &= L(x) &&= x &&= e_2 \\ L(e_3) &= L(x^2) &&= x &&= e_2 \end{aligned} \qquad \underset{e\leftarrow e}{A} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

c. (10 pts) Find the change of basis matrices

- $\underset{e\leftarrow f}{C}$ from the f-basis to the e-basis, and

$$\begin{aligned} f_1 &= 1+x &&= e_1+e_2 \\ f_2 &= x &&= e_2 \\ f_3 &= -x+x^2 &&= -e_2+e_3 \end{aligned} \qquad \underset{e\leftarrow f}{C} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

- $\underset{f\leftarrow e}{C}$ from the e-basis to the f-basis.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \; R2 - R1 + R3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \qquad \underset{f\leftarrow e}{C} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

d. (10 pts) Find the matrix of L relative to the f-basis. Call it $\underset{f\leftarrow f}{B}$.

$$\underset{f\leftarrow f}{B} = \underset{f\leftarrow e}{C}\;\underset{e\leftarrow e}{A}\;\underset{e\leftarrow f}{C} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

e. (5 pts) Find $\underset{f\leftarrow f}{B}$ by a second method.

Recall $L(p) = 2p(0) + p(1)x$

$$\begin{aligned} L(f_1) &= L(1+x) &&= 2+2x &&= 2f_1 \\ L(f_2) &= L(x) &&= x &&= f_2 \\ L(f_3) &= L(-x+x^2) &&= 0 &&= 0 \end{aligned} \qquad \underset{f\leftarrow f}{B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

f. (5 pts Extra Credit) What are the eigenvalues and corresponding eigenpolynomials of L? (This required no computations.)

Since $L(f_1) = 2f_1$, we conclude f_1 is an eigenvector with eigenvalue 2.
Since $L(f_2) = f_2$, we conclude f_2 is an eigenvector with eigenvalue 1.
Since $L(f_3) = 0$, we conclude f_3 is an eigenvector with eigenvalue 0.

3. (40 points) Consider the matrix $A = \begin{pmatrix} -6 & -8 \\ 4 & 6 \end{pmatrix}$.

a. (15 pts) Find the eigenvalues and eigenvectors of A.

$$\det(A - \lambda\mathbf{1}) = \begin{vmatrix} -6-\lambda & -8 \\ 4 & 6-\lambda \end{vmatrix} = (-6-\lambda)(6-\lambda) + 32 = \lambda^2 - 4 \quad \Rightarrow \quad \lambda = 2, -2$$

$$\lambda = 2: \quad \begin{pmatrix} -8 & -8 & 0 \\ 4 & 4 & 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda = -2: \quad \begin{pmatrix} -4 & -8 & 0 \\ 4 & 8 & 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2t \\ t \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

b. (10 pts) Find the diagonalizing matrix X so that $A = XDX^{-1}$ where D is diagonal. What are D and X^{-1}?

$$X = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \qquad X^{-1} = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

c. (5 pts) Find A^{10}.

$$A^{10} = XD^{10}X^{-1} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 2^{10} & 0 \\ 0 & 2^{10} \end{pmatrix}$$

d. (5 pts) Find e^{At}.

$$e^{At} = Xe^{Dt}X^{-1} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} & 2e^{2t} \\ -e^{-2t} & -e^{-2t} \end{pmatrix} = \begin{pmatrix} -e^{2t} + 2e^{-2t} & -2e^{2t} + 2e^{-2t} \\ e^{2t} - e^{-2t} & 2e^{2t} - e^{-2t} \end{pmatrix}$$

e. (5 pts Extra Credit) Find $\sin\left(\frac{\pi}{4}A\right)$.

HINT: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$$\sin\left(\frac{\pi}{4}A\right) = X\sin\left(\frac{\pi}{4}D\right)X^{-1} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \sin\left(\frac{\pi}{4}2\right) & 0 \\ 0 & \sin\left(-\frac{\pi}{4}2\right) \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 2 & 3 \end{pmatrix}$$

4. (10 points) Let $V = \text{Span}\{\vec{v}_1, \vec{v}_2\}$ where

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Notice that V is a subspace of $\mathbf{R}^5$.

a. (6 pts) Find $V^\perp$ the orthogonal subspace to V.

$V^\perp = \left\{\vec{x} \in \mathbf{R}^5 \mid \vec{v}_1 \bullet \vec{x} = 0 \text{ and } \vec{v}_2 \bullet \vec{x} = 0\right\}$ Let $\vec{x} = (a,b,c,d,e)^{\mathbf{T}}$

$\vec{v}_1 \bullet \vec{x} = a + c + e = 0$ $\quad \vec{v}_2 \bullet \vec{x} = b + d = 0$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{Already reduced.} \quad \vec{x} = \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = \begin{pmatrix} -r-t \\ -s \\ r \\ s \\ t \end{pmatrix}$$

$$V^\perp = \left\{ \vec{x} = r\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \text{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

b. (2 pts) What is a basis for $V^\perp$?

$$\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

c. (2 pts) What is the dimension of $V^\perp$?

$\dim V^\perp = 3$