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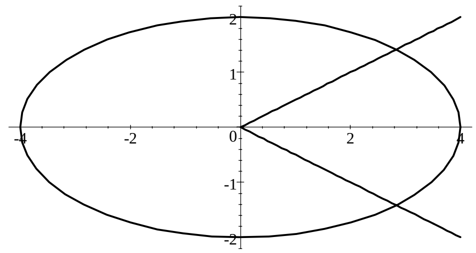
1	/15	3	/15
2	/30	4	/45

MATH 311 Exam 3 Fall 2000
Section 502 Solutions P. Yasskin

1. (15 points) Compute $\iint\limits_R x\,dx\,dy$ over the region inside the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{4}=1$ between the lines $y=\dfrac{x}{2}$ and $y=-\dfrac{x}{2}$ in the 1st and 4th quadrants.

HINT: Use the elliptic coordinate system

$$x=4t\cos\theta,\quad y=2t\sin\theta.$$



The Jacobian is

$$J=\left|\begin{vmatrix} 4\cos\theta & -4t\sin\theta \\ 2\sin\theta & 2t\cos\theta \end{vmatrix}\right| = \left|8t\cos^2\theta+8t\sin^2\theta\right| = 8t$$

The limits are $0\le t\le 1$ and

$$y=\pm\frac{x}{2} \quad\Rightarrow\quad 2t\sin\theta=\pm\frac{1}{2}4t\cos\theta \quad\Rightarrow\quad \tan\theta=\pm1 \quad\Rightarrow\quad \theta=\pm\frac{\pi}{4}$$

So the integral becomes

$$\iint\limits_R x\,dx\,dy = \int_{-\pi/4}^{\pi/4}\int_0^1 4t\cos\theta\,8t\,dt\,d\theta = \left[32\frac{t^3}{3}\right]_0^1\left[\sin\theta\right]_{-\pi/4}^{\pi/4}$$

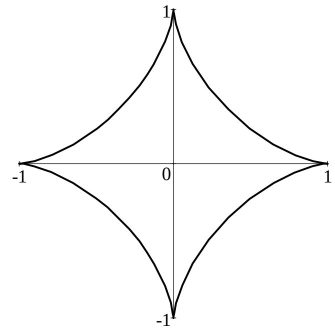
$$=\frac{32}{3}\left[\frac{1}{\sqrt{2}}-\frac{-1}{\sqrt{2}}\right]=\frac{32}{3}\sqrt{2}$$

2. (30 points) The diamond shaped region at the right is called an astroid. It is the graph of

$$x^{2/3} + y^{2/3} = 1$$

and it may be parametrized by

$$x = \cos^3 t \qquad y = \sin^3 t.$$



a. Find its arc length. HINT: First find one quarter of the length.

$$\frac{dx}{dt} = -3\cos^2 t \sin t \qquad \frac{dy}{dt} = 3\sin^2 t \cos t$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(-3\cos^2 t \sin t)^2 + \left(3\sin^2 t \cos t\right)^2}\, dt$$

$$= \sqrt{9\left(\cos^4 t \sin^2 t + \sin^4 t \cos^2 t\right)}\, dt = \sqrt{9\left(\cos^2 t \sin^2 t\right)\left(\cos^2 t + \sin^2 t\right)}\, dt$$

$$= 3\cos t \sin t\, dt$$

$$\frac{1}{4}L = \int_0^{\pi/2} ds = \int_0^{\pi/2} 3\cos t \sin t\, dt = 3\left[\frac{\sin^2 t}{2}\right]_0^{\pi/2} = \frac{3}{2}$$

$$L = 4 \bullet \frac{3}{2} = 6$$

b. Find its area. HINT: What does Green's Theorem say about the integral $\oint(-y\,dx + x\,dy)$?

On the one hand

$$\oint(-y\,dx + x\,dy) = \oint\left(-y\frac{dx}{dt} + x\frac{dy}{dt}\right)dt$$

$$= \int_0^{2\pi}\left(\sin^3 t \bullet 3\cos^2 t \sin t + \cos^3 t \bullet 3\sin^2 t \cos t\right)dt$$

$$= 3\int_0^{2\pi}\left(\sin^4 t \cos^2 t + \cos^4 t \sin^2 t\right)dt$$

$$= 3\int_0^{2\pi} \sin^2 t \cos^2 t\, dt = \frac{3}{4}\int_0^{2\pi} \sin^2(2t)\, dt$$

$$= \frac{3}{8}\int_0^{2\pi}(1 - \cos(4t))\, dt = \frac{3}{8} \bullet 2\pi = \frac{3\pi}{4}$$

On the other hand, by Green's Theorem,

$$\oint(-y\,dx + x\,dy) = \iint\left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y}\right)dx\,dy$$

$$= 2\iint 1\,dx\,dy = 2 \bullet Area$$

So

$$Area = \frac{1}{2}\oint(-y\,dx + x\,dy) = \frac{1}{2} \bullet \frac{3\pi}{4} = \frac{3\pi}{8}$$

3. (15 points) Compute $\int \vec{F} \bullet d\vec{s}$ for $\vec{F} = (z^2 - (\sin x)e^{\cos x}, (\cos y)e^{\sin y}, 2xz)$ along the helix $\vec{r}(\theta) = (\cos\theta, \sin\theta, \theta)$ for $0 \le \theta \le 2\pi$.

HINT: Check $\vec{F}$ is a conservative vector field. Then find a potential OR change the path.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ z^2 - (\sin x)e^{\cos x} & (\cos y)e^{\sin y} & 2xz \end{vmatrix} = \hat{i}(0-0) - \hat{j}(2z - 2z) + \hat{k}(0-0) = (0,0,0)$$

So $\vec{F}$ is conservative and the integral is path independent.

METHOD I: Find a potential:

$$\partial_x f = z^2 - (\sin x)e^{\cos x} \quad \Rightarrow \quad f = xz^2 + e^{\cos x} + g(y,z)$$

$$\partial_y f = (\cos y)e^{\sin y} = \partial_y g \quad \Rightarrow \quad g = e^{\sin y} + h(z)$$

$$\partial_z f = 2xz = 2xz + \partial_z h \quad \Rightarrow \quad h = C \quad \Rightarrow \quad f = xz^2 + e^{\cos x} + e^{\sin y} + C$$

$$\int \vec{F} \bullet d\vec{s} = f(\vec{r}(2\pi)) - f(\vec{r}(0)) = f(1,0,2\pi) - f(1,0,0)$$

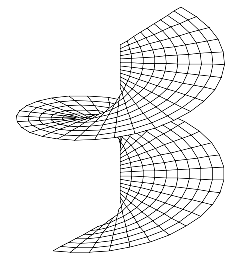
$$= \left[1(2\pi)^2 + e^{\cos 1} + e^{\sin 0}\right] - \left[1(0)^2 + e^{\cos 1} + e^{\sin 0}\right] = 4\pi^2$$

METHOD II: Change the path:

The endpoints are $(1,0,0)$ and $(1,0,2\pi)$. So consider the path $\vec{r}(t) = (1,0,t)$ for $0 \le t \le 2\pi$. The velocity is $\vec{v} = (0,0,1)$. So

$$\int \vec{F} \bullet d\vec{s} = \int \vec{F} \bullet \vec{v}\, dt = \int_0^{2\pi} F_z\, dt = \int_0^{2\pi} 2xz\, dt = \int_0^{2\pi} 2xz\, dt = \int_0^{2\pi} 2t\, dt = \left[t^2\right]_0^{2\pi} = 4\pi^2$$

4. (45 points) The spiral ramp at the right may be parametrized as



$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$$

for $0 \le r \le 2$ and $0 \le \theta \le 3\pi$.

a. Find the tangent plane at the point $(x,y,z) = (-1,0,\pi)$. Give its equation in both parametric form and non-parametric form.

$$\vec{R}_r = (\cos\theta, \sin\theta, 0) \qquad \vec{R}_\theta = (-r\sin\theta, r\cos\theta, 1) \qquad \vec{N} = \vec{R}_r \times \vec{R}_\theta = (\sin\theta, -\cos\theta, r)$$

The point $P = (-1,0,\pi)$ has parameters $\theta = \pi$ and $r = 1$. At this point,

$$\vec{R}_r = (-1,0,0) \qquad \vec{R}_\theta = (0,-1,1) \qquad \vec{N} = \vec{R}_r \times \vec{R}_\theta = (0,1,1)$$

The parametric equation is

$$(x,y,z) = P + s\vec{R}_r + t\vec{R}_\theta = (-1,0,\pi) + s(-1,0,0) + t(0,-1,1) = (-1-s, -t, \pi+t)$$

or

$$x = -1-s \qquad y = -t \qquad z = \pi + t$$

The non-parametric equation is

$$\vec{N} \bullet X = \vec{N} \bullet P \quad \Rightarrow \quad (0,1,1) \bullet (x,y,z) = (0,1,1) \bullet (-1,0,\pi) \quad \Rightarrow \quad y + z = \pi$$

b. Compute $\iint \sqrt{x^2+y^2}\, dS$ over this spiral ramp.

On the spiral ramp,

$$\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r \quad \text{and} \quad \left|\vec{N}\right| = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{1+r^2}$$

So

$$\iint \sqrt{x^2+y^2}\, dS = \int_0^{3\pi}\int_0^2 r\sqrt{1+r^2}\, dr\, d\theta = 3\pi \frac{(1+r^2)^{3/2}}{3}\Bigg|_0^2 = \pi(5^{3/2}-1)$$

c. Compute $\iint \vec{F} \bullet d\vec{S}$ over this spiral ramp if $\vec{F} = (xz, yz, z^2)$.

On the spiral ramp,

$$\vec{F} = (r\theta\cos\theta, r\theta\sin\theta, \theta^2) \quad \text{and} \quad \vec{N} = (\sin\theta, -\cos\theta, r)$$

So

$$\vec{F} \bullet \vec{N} = r\theta\cos\theta\sin\theta - r\theta\sin\theta\cos\theta + r\theta^2 = r\theta^2$$

and

$$\iint \vec{F} \bullet d\vec{S} = \iint \vec{F} \bullet \vec{N}\, dr\, d\theta = \int_0^{3\pi}\int_0^2 r\theta^2\, dr\, d\theta = \left[\frac{\theta^3}{3}\right]_0^{3\pi}\left[\frac{r^2}{2}\right]_0^2 = 18\pi^3$$