Name________________________ ID____________

Math 311 Exam 2 Spring 2002

Section 503 P. Yasskin

Solutions

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|---|---|---|---|
| 1 | /10 | 5 | /10 |
| 2 | /15 | 6 | /15 |
| 3 | /10 | 7 | /25 |
| 4 | /15 | | |

**1.** (10 points) Which one of the following is **NOT** a vector space? Why?

**a.** $Q = \{(w,x,y,z) \in \mathbf{R}^4 \mid w + 2x + 3y + 4z = 0\}$

**b.** $S = \{X \in M(2,2) \mid AXA = X\}$ where $M(2,2)$ is the set of $2 \times 2$ matrices and $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

**c.** $T = \{p \in P_3 \mid p(1) = p(0) + 1\}$ where $P_3$ is the set of polynomials of degree less than 3.

$T$ is NOT a vector space.

$$\begin{aligned} p,q \in T \quad &\Rightarrow \quad p(1) = p(0) + 1 \text{ and } q(1) = q(0) + 1 \\ &\Rightarrow \quad (p+q)(1) = p(1) + q(1) = p(0) + 1 + q(0) + 1 = (p+q)(0) + 2 \neq (p+q)(0) + 1 \\ &\Rightarrow \quad p + q \notin T \end{aligned}$$

**2.** (15 points) For **ONE** of the two vector spaces listed in #1 (say which), give a basis and the dimension.

$Q$ is a vector space which is a 3-plane in $R^4$ thru the origin. The augmented matrix of the equation is $(1 \;\; 2 \;\; 3 \;\; 4 \;\; | \;\; 0)$. So the parametric solution is

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2r - 3s - 4t \\ r \\ s \\ t \end{pmatrix} = r\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Basis is $\begin{pmatrix} -2 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -4 & 0 & 0 & 1 \end{pmatrix}$ $\quad \dim Q = 3$

OR

$S$ is a vector space.

If $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $AXA = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -d & c \\ b & -a \end{pmatrix}$

So $X \in S$ means $\begin{pmatrix} -d & c \\ b & -a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or $d = -a$ and $c = b$.

So $X = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Basis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \dim S = 2$

**3.** (10 points) Which one of the following is NOT a linear function? Why?

**a.** $I : C[0,2\pi] \to \mathbf{R}$ where $C[0,2\pi]$ is the set of continuous real valued functions on the interval $[0,2\pi]$ and $I(f) = \int_0^{2\pi} [x + f(x)]\,dx$.

**b.** $Z : M(2,2) \to M(2,2)$ given by $Z(X) = BXB$ where $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

**c.** $E : P_3 \to P_3$ given by $E(p(x)) = x\dfrac{dp(x)}{dx} - p(x)$.

$I$ is NOT linear because

$I(af) = \int_0^{2\pi} [x + af(x)]\,dx$ but $aI(f) = a\int_0^{2\pi} [x + f(x)]\,dx = \int_0^{2\pi} [ax + af(x)]\,dx$

which are not equal.

**4.** (15 points) For **ONE** of the two linear functions listed in #3 (say which), find the kernel and the image (as the span of some vectors) and determine if the function is onto and/or one-to-one.

$Z$ is linear. If $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $Z(X) = BXB = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+c & a+c \\ 0 & 0 \end{pmatrix}$.

$$Ker(Z) = \{X \mid Z(X) = \mathbf{0}\} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + c = \mathbf{0}\right\} = \left\{\begin{pmatrix} a & b \\ -a & d \end{pmatrix}\right\}$$

$$= \left\{a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\} = Span\left\{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

$$Im(Z) = \{Z(X)\} = \left\{\begin{pmatrix} a+c & a+c \\ 0 & 0 \end{pmatrix}\right\} = \left\{(a+c)\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right\} = Span\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right\}$$

$Z$ is NOT onto because $Im(Z) \neq M(2,2)$. $Z$ is NOT one-to-one because $Ker(Z) \neq \{\mathbf{0}\}$.

OR

$E$ is linear. If $p(x) = a + bx + cx^2$, then $E(p(x)) = x(b + 2cx) - (a + bx + cx^2) = -a + cx^2$.

$Ker(E) = \{p \mid E(p) = 0\} = \{a + bx + cx^2 \mid -a + cx^2 = 0\} = \{a + bx + cx^2 \mid a = c = 0\}$

$= \{bx\} = Span\{x\}$

$Im(E) = \{E(p)\} = \{-a + cx^2\} = Span\{1, x\}$

$E$ is NOT onto because $Im(Z) = Span\{1, x\} \neq P_3$. $E$ is NOT one-to-one because $Ker(E) \neq \{0\}$.

**5.** (10 points) Let $P_3$ be the vector space of polynomials of degree less than 3. Which one of the following is NOT an inner product on $P_3$? Why? HINT: Let $p = a + bx + cx^2$ and find which is not positive definite.

**a.** $\langle p,q\rangle_1 = \int_0^1 x p(x) q(x)\, dx$

**b.** $\langle p,q\rangle_2 = p(0)q(0) + p(1)q(1)$

**c.** $\langle p,q\rangle_3 = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$

$\langle p,q\rangle_2$ is NOT an inner product, because if $p = a + bx + cx^2$ then
$\langle p,p\rangle_2 = p(0)^2 + p(1)^2 = (a)^2 + (a+b+c)^2$
So if $p = x - x^2$ then $a = 0, b = 1, c = -1$ and $\langle p,p\rangle_2 = (0)^2 + (0+1-1)^2 = 0$, but $p \neq 0$

**6.** (15 points) For **ONE** of the two inner products listed in #5 (say which), find the angle between the polynomials

$$r(x) = 4x \qquad \text{and} \qquad s(x) = 6x^2$$

$\langle p,q\rangle_1 = \int_0^1 x p(x) q(x)\, dx$ is an inner product.

$\langle r,r\rangle_1 = \int_0^1 x r(x)^2\, dx = \int_0^1 16x^3\, dx = [4x^4]_0^1 = 4 \qquad |r| = \sqrt{\langle r,r\rangle_1} = 2$

$\langle s,s\rangle_1 = \int_0^1 x s(x)^2\, dx = \int_0^1 36x^5\, dx = [6x^6]_0^1 = 6 \qquad |s| = \sqrt{\langle s,s\rangle_1} = \sqrt{6}$

$\langle r,s\rangle_1 = \int_0^1 x r(x) s(x)\, dx = \int_0^1 24x^4\, dx = \left[24\frac{x^5}{5}\right]_0^1 = \frac{24}{5}$

$\cos\theta = \frac{\langle r,s\rangle_1}{|r||s|} = \frac{24}{5 \cdot 2 \cdot \sqrt{6}} = \frac{2}{5}\sqrt{6} \qquad \Rightarrow \qquad \theta = \cos^{-1}\frac{2\sqrt{6}}{5} \approx 0.201 \text{ rad}$

OR

$\langle p,q\rangle_3 = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$ is an inner product.

$\langle r,r\rangle_3 = \langle 4x, 4x\rangle_3 = 16 + 0 + 16 = 32 \qquad |r| = \sqrt{\langle r,r\rangle_3} = \sqrt{32} = 4\sqrt{2}$

$\langle s,s\rangle_3 = \langle 6x^2, 6x^2\rangle_3 = 36 + 0 + 36 = 72 \qquad |s| = \sqrt{\langle s,s\rangle_3} = \sqrt{72} = 6\sqrt{2}$

$\langle r,s\rangle_3 = \langle 4x, 6x^2\rangle_3 = -24 + 0 + 24 = 0$

$\cos\theta = \frac{\langle r,s\rangle_3}{|r||s|} = \frac{0}{4\sqrt{2}\,6\sqrt{2}} = 0 \qquad \Rightarrow \qquad \theta = \cos^{-1}0 \approx \frac{\pi}{2} \text{ rad}$

**7.** (25 points) Consider the vector space $V = Span\{\sin^2\theta, \cos^2\theta, \sin\theta\cos\theta\}$

**a.** (3) Show $e_1 = \sin^2\theta$, $e_2 = \cos^2\theta$, $e_3 = \sin\theta\cos\theta$ is a basis for $V$.

By definition of $V$, $e_1$, $e_2$ and $e_3$ span $V$. Are they independent?

$ae_1 + be_2 + ce_3 = 0$

$a\sin^2\theta + b\cos^2\theta + c\sin\theta\cos\theta = 0$

$\theta = 0: \quad \Rightarrow \quad b = 0$

$\theta = \frac{\pi}{2}: \quad \Rightarrow \quad a = 0$

$\theta = \frac{\pi}{4}: \quad \Rightarrow \quad \frac{a}{2} + \frac{b}{2} + \frac{c}{2} = 0 \quad \Rightarrow \quad c = 0$

So they are independent and they are a basis.

**b.** (7) Another basis is $f_1 = 1$, $f_2 = \sin 2\theta$, $f_3 = \cos 2\theta$. Find the change of basis matrices from the $e$-basis to the $f$-basis and vice versa. Be sure to say which is which.

$$
\begin{aligned}
f_1 &= 1 &&= \sin^2\theta + \cos^2\theta &&= e_1 + e_2 \\
f_2 &= \sin 2\theta &&= 2\sin\theta\cos\theta &&= 2e_3 \\
f_3 &= \cos 2\theta &&= \cos^2\theta - \sin^2\theta &&= -e_1 + e_2
\end{aligned}
\qquad
\underset{e\leftarrow f}{C} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}
$$

$$
\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right) R_2 - R_1 \quad \Rightarrow \quad
\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \frac{1}{2}R_3 \\ \frac{1}{2}R_2 \end{array} \quad \Rightarrow
$$

$$
\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & 0 \end{array}\right) R_1 + R_3 \quad \Rightarrow \quad
\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{2} & \frac{1}{2} & 0 \end{array}\right)
$$

$$
\underset{f\leftarrow e}{C} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & 0 \end{pmatrix}
$$

(7 continued)

**c.** (4) Consider the derivative operator $D : V \to V$ given by $D(g) = \dfrac{dg}{d\theta}$.
Find the matrix of $D$ relative to the $e$-basis.

$$\begin{aligned} D(e_1) &= D(\sin^2\theta) &&= 2\sin\theta\cos\theta &&= 2e_3 \\ D(e_2) &= D(\cos^2\theta) &&= -2\sin\theta\cos\theta &&= -2e_3 \\ D(e_3) &= D(\sin\theta\cos\theta) &&= \cos^2\theta - \sin^2\theta &&= -e_1 + e_2 \end{aligned} \qquad \underset{e\leftarrow e}{A} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix}$$

**d.** (6) Find the matrix of $D$ relative to the $f$-basis in **TWO** ways.

**i.** by using the matrix of $D$ relative to the $e$-basis:

$$\underset{f\leftarrow f}{A} = \underset{f\leftarrow e}{C}\ \underset{e\leftarrow e}{A}\ \underset{e\leftarrow f}{C} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

**ii.** by differentiating basis vectors:

$$\begin{aligned} D(f_1) &= D(1) &&= 0 &&= 0 \\ D(f_2) &= D(\sin 2\theta) &&= 2\cos 2\theta &&= 2f_3 \\ D(f_3) &= D(\cos 2\theta) &&= -2\sin 2\theta &&= -2f_2 \end{aligned} \qquad \underset{f\leftarrow f}{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

**e.** (5) Consider the function $g(\theta) = 5\sin 2\theta + 3\cos 2\theta$. Compute $D(g)$ in **TWO** ways:

**i.** by differentiating:

$$D(g) = D(5\sin 2\theta + 3\cos 2\theta) = 10\cos 2\theta - 6\sin 2\theta$$

**ii.** by using the matrix of $D$ relative to the $f$-basis:

$$(g)_f = \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix} \qquad [D(g)]_f = \underset{f\leftarrow f}{A}\ (g)_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ 10 \end{pmatrix}$$

$$D(g) = -6f_2 + 10f_3 = -6\sin 2\theta + 10\cos 2\theta$$