Name________________________________

Math 311	Exam 1	Spring 2010	
Section 502	Solutions	P. Yasskin	

1	/25	4	/10
2	/25	5	/20
3	/25	Total	/105

1. (25 points) Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 3 \end{pmatrix} \qquad X = \begin{pmatrix} a & p \\ b & q \\ c & r \\ d & s \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$$

a. Compute A^{-1}.

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 3 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ R_3 - R_1 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4 - 2R_2 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & -2 & 0 & 1 \end{array}\right) \begin{array}{l} \\ \\ R_4 \\ R_3 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2 - R_3 \\ \\ -R_4 \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & -2 & 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -3 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 3 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} R_1 + 2R_4 \\ R_2 + 3R_4 \\ R_3 - 3R_4 \\ \\ \end{array}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & -2 & -1 \\ 0 & 1 & 0 & 0 & 3 & 3 & -3 & -1 \\ 0 & 0 & 1 & 0 & -3 & -2 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & 2 & -2 & -1 \\ 3 & 3 & -3 & -1 \\ -3 & -2 & 3 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

b. Solve the equation $AX = B$.

$$X = A^{-1}B = \begin{pmatrix} 3 & 2 & -2 & -1 \\ 3 & 3 & -3 & -1 \\ -3 & -2 & 3 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ -1 & -3 \\ 0 & 2 \end{pmatrix}$$

2. (25 points) Consider the system of equations:

$$\begin{aligned} 3x + 5y - 2z &= 17 \\ x + y \quad &= 5 \\ 3y - 3z &= p \end{aligned}$$

a. Write out the augmented matrix and row reduce it to reduced row eschelon form. (Give reasons for each step.)

$$\left(\begin{array}{ccc|c} 3 & 5 & -2 & 17 \\ 1 & 1 & 0 & 5 \\ 0 & 3 & -3 & p \end{array}\right) \begin{array}{l} R_2 \\ R_1 \\ \\ \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 3 & -3 & p \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 3R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 3 & 5 & -2 & 17 \\ 0 & 3 & -3 & p \end{array}\right) \begin{array}{l} \\ R_2 - 3R_1 \\ \\ \end{array} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & p-3 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \\ \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 2 & -2 & 2 \\ 0 & 3 & -3 & p \end{array}\right) \begin{array}{l} \\ \frac{1}{2}R_2 \\ \\ \end{array}$$

b. For what value(s) of p are there (At least one answer is "No p".)

no solutions? $p \neq 3$

a unique solution? No p

exactly two solutions? No p

infinitely-many solutions? $p = 3$

c. For those p's for which there are solutions, what are the solutions?

For $p = 3$, the equations are:

$$\begin{aligned} x + z &= 4 \\ y - z &= 1 \end{aligned}.$$

Since z is a free variable, the solutions are:

$$\begin{aligned} x &= 4 - r \\ y &= 1 + r \\ z &= r \end{aligned}$$

3. (25 points) Consider the system of equations:

$$\begin{aligned} 3x + 5y - 2z &= 17 \\ x + y \quad &= 5 \\ 2y - z &= p \end{aligned}$$

a. Write out the augmented matrix and row reduce it to reduced row eschelon form. (Give reasons for each step.)

$$\left(\begin{array}{ccc|c} 3 & 5 & -2 & 17 \\ 1 & 1 & 0 & 5 \\ 0 & 2 & -1 & p \end{array}\right) \begin{array}{l} R_2 \\ R_1 \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 3 & 5 & -2 & 17 \\ 0 & 2 & -1 & p \end{array}\right) \begin{array}{l} \\ R_2 - 3R_1 \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 2 & -2 & 2 \\ 0 & 2 & -1 & p \end{array}\right) \begin{array}{l} \\ \frac{1}{2}R_2 \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -1 & p \end{array}\right) \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 2R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & p-2 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \\ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 6-p \\ 0 & 1 & 0 & p-1 \\ 0 & 0 & 1 & p-2 \end{array}\right)$$

b. For what value(s) of p are there (At least one answer is "No p".)

no solutions? No p

a unique solution? All p

exactly two solutions? No p

infinitely-many solutions? No p

c. For those p's for which there are solutions, what are the solutions?

$$\begin{aligned} x &= 6 - p \\ y &= p - 1 \\ z &= p - 2 \end{aligned}$$

4. (10 points) Write out the equations which need to be solved to find a, b, c and d to balance the chemical equation:

$$a\,\mathrm{Ca\,O_2\,H_2} + b\,\mathrm{H_3\,P\,O_4} \Rightarrow c\,\mathrm{Ca_3\,P_2\,O_8} + d\,\mathrm{H_2\,O}$$

Then set up the augmented matrix for the equations. Do not solve the equations.

Ca:	$a = 3c$	$a - 3c = 0$	
O:	$2a + 4b = 8c + d$	$2a + 4b - 8c - d = 0$	
H:	$2a + 3b = 2d$	$2a + 3b - 2d = 0$	
P:	$b = 2c$	$b - 2c = 0$	

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 2 & 4 & -8 & -1 & 0 \\ 2 & 3 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 & 0 \end{array}\right)$$

5. (20 points) Say whether each of the following statements is true or false for all scalars a, b and c, all 3×3 matrices A, B and C and all 3×3 elementary matrices E_1, E_2 and E_3 of types I, II and III respectively where E_1, E_2 and E_3 are NOT the unit matrix. Circle your answers.

a. $(A+B)C = AC + BC$........ **True** False

b. $(A+B)^2 = A^2 + 2AB + B^2$........ True **False**

c. $(AB)^2 = A^2B^2$........ True **False**

d. If $B = A - A^{\mathsf{T}}$ then $B^{\mathsf{T}} = -B$........ **True** False

e. $(AB)^{\mathsf{T}} = A^{\mathsf{T}}B^{\mathsf{T}}$........ True **False**

f. $(AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}}$........ **True** False

g. $(AB)^{-1} = A^{-1}B^{-1}$........ True **False**

h. $(AB)^{-1} = B^{-1}A^{-1}$........ **True** False

i. $(aA + bB)^{\mathsf{T}} = \left(aA^{\mathsf{T}} + bB^{\mathsf{T}}\right)$........ **True** False

j. $(aA + bB)^{-1} = (aA^{-1} + bB^{-1})$........ True **False**

k. $E_1 A = AE_1^{\mathsf{T}}$........ True **False**

l. $E_2 A = A^{\mathsf{T}}E_2^{\mathsf{T}}$........ True **False**

m. $E_3 A = \left(A^{\mathsf{T}}E_3^{\mathsf{T}}\right)^{\mathsf{T}}$........ **True** False

n. $\det(cA) = c^3 \det A$........ **True** False

o. $\det(AB^{-1}) = \det A - \det B$........ True **False**

p. $\det\left(AB^{\mathsf{T}}\right) = \det A \det B$........ **True** False

q. $\det(E_1 A) = \dfrac{1}{\det A}$........ True **False**

r. $\det(E_2 A) = k \det A$ with $k \neq 1$........ **True** False

s. $\det(E_3 A) = k \det A$ with $k \neq 1$........ True **False**

t. $\det(E_1 E_2 E_3 A E_3^{-1} E_2^{-1}) = -\det A$........ **True** False