1. (45 points) Consider the vector space of functions V = Span(1, cos x, cos² x) with basis

\[ v_1 = 1 \quad v_2 = \cos x \quad v_3 = \cos^2 x \]

a. (5 pts) Show \( \cos 2x \in V \).

HINT: Use a trig identity.

b. (10 pts) Show another basis for \( V \) is

\[ p_1 = 1 \quad p_2 = \cos x \quad p_3 = \cos 2x \]

c. (6 pts) Find the change of basis matrix from the \( p \) basis to the \( v \) basis. Call it \( C_{v-p} \).

d. (6 pts) Find the change of basis matrix from the \( v \) basis to the \( p \) basis. Call it \( C_{p-v} \).
e. (6 pts) Consider the linear function $L : V \to \mathbb{R}^2$ given by $L(f) = \begin{pmatrix} f(0) \\ f\left(\frac{\pi}{2}\right) \end{pmatrix}$.

Find the matrix of $L$ relative to the $v$ basis for $V$ and the basis $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $\mathbb{R}^2$. Call it $A_v$.

f. (6 pts) Find the matrix of $L$ relative to the $p$ basis for $V$ and the $e$ basis for $\mathbb{R}^2$. Call it $B_p$.

g. (6 pts) Recompute $B_p$ by a second method.

(One method should use the change of basis matrix, the other should not.)
2. (40 points) Consider a linear map \( L : \mathbb{R}^n \to \mathbb{R}^p \) whose matrix relative to the standard bases is
\[
A = \begin{pmatrix}
1 & -2 & 0 & -4 \\
-1 & 2 & 1 & 2 \\
0 & 0 & 1 & -2
\end{pmatrix}.
\]

a. (4 pts) What are \( n \) and \( p \)?

b. (9 pts) Identify the kernel of \( L \), a basis for the kernel, and the dimension of the kernel.
c. (9 pts) Identify the image of \( L \), a basis for the image, and the dimension of the image.

d. (6 pts) Is the function \( L \) one-to-one? Why?

e. (6 pts) Is the function \( L \) onto? Why?

f. (6 pts) Verify the dimensions in a, b and c agree with the Nullity-Rank Theorem.
3. (30 points) Consider the vector space of functions \( V = \text{Span}(1, \cos x, \cos^2 x) \)
with the inner product \( \langle f, g \rangle = \int_0^\pi f(x)g(x) \sin x \, dx \).

Here is a table of integrals:

\[
\begin{align*}
\int_0^\pi \sin x \, dx &= 2 & \int_0^\pi \cos x \sin x \, dx &= 0 & \int_0^\pi \cos^2 x \sin x \, dx &= \frac{2}{3} \\
\int_0^\pi \cos^3 x \sin x \, dx &= 0 & \int_0^\pi \cos^4 x \sin x \, dx &= \frac{2}{5} & \int_0^\pi \cos^5 x \sin x \, dx &= 0
\end{align*}
\]

a. (10 pts) Find the angle between the functions \( p = 1 \) and \( q = \cos^2 x \).
b. (20 pts) Apply the Gram-Schmidt procedure to the basis

\[ v_1 = 1 \quad v_2 = \cos x \quad v_3 = \cos^2 x \]

to produce an orthogonal basis \( w_1, w_2, w_3 \) and an orthonormal basis \( u_1, u_2, u_3 \).