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Math 311 Exam 2 Spring 2010
Section 502 Solutions P. Yasskin

1	/45	3	/30
2	/40	Total	/115

1. (45 points) Consider the vector space of functions $V = Span(1, \cos x, \cos^2 x)$ with basis

$$v_1 = 1 \qquad v_2 = \cos x \qquad v_3 = \cos^2 x$$

a. (5 pts) Show $\cos 2x \in V$.
HINT: Use a trig identity.

$\cos 2x = 2\cos^2 x - 1 = -1 \cdot v_1 + 2 \cdot v_3 \in V$.

b. (10 pts) Show another basis for V is

$$p_1 = 1 \qquad p_2 = \cos x \qquad p_3 = \cos 2x$$

Since there are 3 p's and $\dim V = 3$, we only need to show the p's span OR are linearly independent.

To show they span, let $f \in V$. Then

$$f = a \cdot 1 + b \cdot \cos x + c \cdot \cos^2 x = a \cdot 1 + b \cdot \cos x + c \cdot \frac{1+\cos 2x}{2}$$
$$= \left(a + \frac{c}{2}\right) \cdot 1 + b \cdot \cos x + \frac{c}{2} \cdot \cos 2x = \left(a + \frac{c}{2}\right)p_1 + bp_2 + \frac{c}{2}p_3$$

So $\{p_1, p_2, p_3\}$ span V.

To show they are linearly independent, suppose $a \cdot 1 + b \cdot \cos x + c \cdot \cos 2x = 0$. Then

$$\begin{array}{lll} x = 0: & a + b + c = 0 & (1) \\ x = \frac{\pi}{2}: & a - c = 0 & (2) \\ x = \pi: & a - b + c = 0 & (3) \end{array} \Rightarrow \begin{array}{lll} (1) - (3) \Rightarrow & b = 0 & (4) \\ (1) + (3) \Rightarrow & a + c = 0 & (5) \end{array} \Rightarrow \begin{array}{ll} (2) + (5) \Rightarrow & a = 0 \\ (2) - (5) \Rightarrow & c = 0 \end{array}$$

So $\{p_1, p_2, p_3\}$ are linearly independent. (OR expand in v's.)

c. (6 pts) Find the change of basis matrix from the p basis to the v basis. Call it $\underset{v \leftarrow p}{C}$.

$$\begin{array}{l} p_1 = 1 = v_1 \\ p_2 = \cos x = v_2 \\ p_3 = \cos 2x = -1 \cdot v_1 + 2 \cdot v_3 \end{array} \Rightarrow \underset{v \leftarrow p}{C} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

d. (6 pts) Find the change of basis matrix from the v basis to the p basis. Call it $\underset{p \leftarrow v}{C}$.

$$\begin{array}{l} v_1 = 1 = p_1 \\ v_2 = \cos x = p_2 \\ v_3 = \cos^2 x = \dfrac{1+\cos 2x}{2} = \dfrac{1}{2}p_1 + \dfrac{1}{2}p_3 \end{array} \Rightarrow \underset{p \leftarrow v}{C} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

(OR find the inverse of $\underset{v \leftarrow p}{C}$.)

e. (6 pts) Consider the linear function $L : V \longrightarrow \mathbb{R}^2$ given by $L(f) = \begin{pmatrix} f(0) \\ f\left(\frac{\pi}{2}\right) \end{pmatrix}$.

Find the matrix of L relative to the v basis for V and the basis $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $\mathbb{R}^2$. Call it $\underset{e \leftarrow v}{A}$.

$$L(v_1) = L(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e_1 + e_2$$

$$L(v_2) = L(\cos x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1 \qquad \Rightarrow \quad \underset{e \leftarrow v}{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$L(v_3) = L(\cos^2 x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1$$

f. (6 pts) Find the matrix of L relative to the p basis for V and the e basis for $\mathbb{R}^2$. Call it $\underset{e \leftarrow p}{B}$.

$$L(p_1) = L(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e_1 + e_2$$

$$L(p_2) = L(\cos x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1 \qquad \Rightarrow \quad \underset{e \leftarrow p}{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

$$L(p_3) = L(\cos 2x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = e_1$$

g. (6 pts) Recompute $\underset{e \leftarrow p}{B}$ by a second method.

(One method should use the change of basis matrix, the other should not.)

$$\underset{e \leftarrow p}{B} = \underset{e \leftarrow v}{A} \; \underset{v \leftarrow p}{C} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

2. (40 points) Consider a linear map $L : \mathbf{R}^n \rightarrow \mathbf{R}^p$ whose matrix relative to the standard bases is

$$A = \begin{pmatrix} 1 & -2 & 0 & -4 \\ -1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

a. (4 pts) What are n and p?

$n = 4 \qquad p = 3$

b. (9 pts) Identify the kernel of L, a basis for the kernel, and the dimension of the kernel.

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -4 & 0 \\ -1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 0 & -4 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 0 & -4 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s+4t \\ s \\ 2t \\ t \end{pmatrix} = s\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix} \qquad \Rightarrow Ker(L) = Span\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix}\right\}$$

$$basis = \left\{\begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix}\right\} \qquad \dim Ker(L) = 2$$

c. (9 pts) Identify the image of L, a basis for the image, and the dimension of the image.

$$Im(L) = Span\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} \right\}$$

$$= Span\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} \right\}$$ Are they independent?

$$a\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 4t \\ 2t \\ t \end{pmatrix}$$

Not independent. Throw away the 3rd vector because c is the free variable.

$$Im(L) = Span\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \quad basis = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \dim Im(L) = 2$$

d. (6 pts) Is the function L one-to-one? Why?

L is not $1-1$ because $Ker(L) \neq \{0\}$

e. (6 pts) Is the function L onto? Why?

L is not onto because $\dim Im(L) = 2$ but $\dim Codom(L) = 3$

f. (6 pts) Verify the dimensions in a, b and c agree with the Nullity-Rank Theorem.

$$\dim Ker(L) + \dim Im(L) = 2 + 2 = 4 = \dim Dom(L)$$

3. (30 points) Consider the vector space of functions $V = Span(1, \cos x, \cos^2 x)$ with the inner product $\langle f, g\rangle = \int_0^\pi f(x)g(x)\sin x\,dx$.

Here is a table of integrals:

$$\int_0^\pi \sin x\,dx = 2 \qquad \int_0^\pi \cos x \sin x\,dx = 0 \qquad \int_0^\pi \cos^2 x \sin x\,dx = \frac{2}{3}$$

$$\int_0^\pi \cos^3 x \sin x\,dx = 0 \qquad \int_0^\pi \cos^4 x \sin x\,dx = \frac{2}{5} \qquad \int_0^\pi \cos^5 x \sin x\,dx = 0$$

a. (10 pts) Find the angle between the functions $p = 1$ and $q = \cos^2 x$.

$$|p| = \sqrt{\langle p, p\rangle} = \sqrt{\langle 1, 1\rangle} = \sqrt{\int_0^\pi \sin x\,dx} = \sqrt{2}$$

$$|q| = \sqrt{\langle q, q\rangle} = \sqrt{\langle \cos^2 x, \cos^2 x\rangle} = \sqrt{\int_0^\pi \cos^4 x \sin x\,dx} = \sqrt{\frac{2}{5}}$$

$$\langle p, q\rangle = \langle 1, \cos^2 x\rangle = \int_0^\pi \cos^2 x \sin x\,dx = \frac{2}{3}$$

$$\cos(\theta) = \frac{\langle p, q\rangle}{|p||q|} = \frac{2}{3\sqrt{2}}\sqrt{\frac{5}{2}} = \frac{\sqrt{5}}{3} \qquad \theta = \arccos\frac{\sqrt{5}}{3} \approx 0.73 \text{ rad} \approx 41.8°$$

b. (20 pts) Apply the Gram-Schmidt procedure to the basis

$$v_1 = 1 \qquad v_2 = \cos x \qquad v_3 = \cos^2 x$$

to produce an orthogonal basis w_1, w_2, w_3 and an orthonormal basis u_1, u_2, u_3.

$$w_1 = v_1 = \boxed{1}$$

$$\langle w_1, w_1\rangle = \langle 1, 1\rangle = \int_0^\pi \sin x\,dx = 2 \qquad |w_1| = \sqrt{2}$$

$$\langle v_2, w_1\rangle = \langle \cos x, 1\rangle = \int_0^\pi \cos x \sin x\,dx = 0$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 = v_2 = \boxed{\cos x}$$

$$\langle w_2, w_2\rangle = \langle \cos x, \cos x\rangle = \int_0^\pi \cos^2 x \sin x\,dx = \frac{2}{3} \qquad |w_2| = \sqrt{\frac{2}{3}}$$

$$\langle v_3, w_1\rangle = \langle \cos^2 x, 1\rangle = \int_0^\pi \cos^2 x \sin x\,dx = \frac{2}{3} \qquad \langle v_3, w_2\rangle = \langle \cos^2 x, \cos x\rangle = \int_0^\pi \cos^3 x \sin x\,dx = 0$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 = \cos^2 x - \frac{2}{3\cdot 2} 1 - 0 = \boxed{\cos^2 x - \frac{1}{3}}$$

$$\langle w_3, w_3\rangle = \left\langle \cos^2 x - \frac{1}{3}, \cos^2 x - \frac{1}{3}\right\rangle = \int_0^\pi \left(\cos^2 x - \frac{1}{3}\right)^2 \sin x\,dx$$

$$= \int_0^\pi \left(\cos^2 x - \frac{1}{3}\right)^2 \sin x\,dx = \int_0^\pi \left(\cos^4 x - \frac{2}{3}\cos^2 x + \frac{1}{9}\right) \sin x\,dx$$

$$= \frac{2}{5} - \frac{2}{3}\cdot\frac{2}{3} + \frac{1}{9}\cdot 2 = \frac{8}{45} \qquad |w_3| = \sqrt{\frac{8}{45}}$$

$$u_1 = \frac{w_1}{|w_1|} = \boxed{\frac{1}{\sqrt{2}}} \qquad u_2 = \frac{w_2}{|w_2|} = \boxed{\sqrt{\frac{3}{2}}\cos x} \qquad u_3 = \frac{w_3}{|w_3|} = \boxed{\sqrt{\frac{45}{8}}\left(\cos^2 x - \frac{1}{3}\right)}$$