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Math 311	Exam 3	Spring 2010
Section 502	Solutions	P. Yasskin

1,2	/ 4	10	/15
3,4	/12	11,12	/12
5	/10	13	/15
6,7	/12	14,15	/10
8,9	/16	Total	/106

Consider the vector space P_3 of polynomials with degree less than 3 with standard basis

$$e_1 = 1 \qquad e_2 = x \qquad e_3 = x^2$$

and the linear operator

$$L : P_3 \to P_3 : L(p) = (x-2)\frac{dp}{dx}$$

1. (2 pts) What is $\dim P_3$? What is the size of the matrix A of the linear map L?

$\dim P_3 = 3$ $\qquad$ A is a 3×3 matrix.

2. (2 pts) Let $p = a + bx + cx^2$. Compute $L(p)$.

$$L(p) = (x-2)\frac{dp}{dx} = (x-2)(b+2cx)$$

3. (10 pts) Identify the kernel of L, a basis for the kernel, and the dimension of the kernel.

$$L(p) = (-2b) + (b-4c)x + (2c)x^2 = 0$$
$$\Rightarrow \qquad -2b = 0 \qquad b - 4c = 0 \qquad 2c = 0 \qquad \Rightarrow \qquad b = c = 0 \qquad \Rightarrow \qquad p = a$$
$$Ker(L) = \{p = a\} = \{\text{constant polynomials}\} = Span(1) \qquad basis = \{1\} \qquad \dim Ker(L) = 1$$

4. (2 pts) What does the kernel of L, (found in part 2), say about one of the eigenvalues of L, and the eigenpolynomial(s) for that eigenvalue? No new computations!

$L(1) = 0 \cdot 1$ $\qquad$ So 0 is an eigenvalue and 1 is the eigenpolynomial.

5. (10 pts) Identify the image of L, a basis for the image, and the dimension of the image.

$$L(p) = (-2b) + (b-4c)x + (2c)x^2 = b(-2+x) + c(-4x+2x^2)$$
$$Im(L) = \{L(p)\} = Span(-2+x, -4x+2x^2) \qquad basis = \{-2+x, -4x+2x^2\} \qquad \dim Im(L) = 2$$

6. (6 pts) Is the function L one-to-one? Why?

L is not $1-1$ because $Ker(L) \neq \{0\}$

7. (6 pts) Is the function L onto? Why?

L is not onto because $\dim Im(L) = 2$ but $\dim Codom(L) = \dim P_3 = 3$

8. (6 pts) Find the matrix of the linear map L relative to the e basis. Call it $\underset{e\leftarrow e}{A}$.

$$\begin{array}{lllll} L(e_1) = & L(1) = & = (x-2)(0) & = 0 & = 0 \\ L(e_2) = & L(x) = & = (x-2)(1) & = -2+x & = -2e_1 + e_2 \\ L(e_3) = & L(x^2) = & = (x-2)(2x) & = -4x+2x^2 & = -4e_2 + 2e_3 \end{array} \qquad \underset{e\leftarrow e}{A} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 2 \end{pmatrix}$$

9. (10 pts) Find the eigenvalues of $\underset{e\leftarrow e}{A}$.

$$\det(A - \lambda\mathbf{1}) = \det\begin{pmatrix} -\lambda & -2 & 0 \\ 0 & 1-\lambda & -4 \\ 0 & 0 & 2-\lambda \end{pmatrix} = -\lambda(1-\lambda)(2-\lambda) = 0 \qquad \Rightarrow \qquad \lambda = 0,1,2$$

10. (15 pts) Find the eigenvectors for each eigenvalue. Call them $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

$$A\vec{v} = \lambda\vec{v} \qquad (A - \lambda\mathbf{1})\vec{v} = 0 \qquad \left(\begin{array}{ccc|c} -\lambda & -2 & 0 & 0 \\ 0 & 1-\lambda & -4 & 0 \\ 0 & 0 & 2-\lambda & 0 \end{array}\right)$$

a. $\lambda_1 = 0$:

$$\left(\begin{array}{ccc|c} 0 & -2 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \begin{array}{l} R_2 \\ \frac{1}{2}R_3 \\ R_1+2R_2 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right) \begin{array}{l} R_1+4R_2 \\ \\ R_3+8R_2 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{l} y = 0 \\ z = 0 \\ x = r \end{array} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} = r\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

b. $\lambda_2 = 1$:

$$\left(\begin{array}{ccc|c} -1 & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \begin{array}{l} -R_1 \\ R_3 \\ R_2+4R_3 \end{array} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{l} x + 2y = 0 \\ z = 0 \\ y = r \end{array} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2r \\ r \\ 0 \end{pmatrix} = r\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

c. $\lambda_3 = 2$:

$$\left(\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} -\frac{1}{2}R_1 \\ -R_2 \\ \end{array} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \begin{array}{l} R_1-R_2 \\ \\ \end{array} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{l} x - 4z = 0 \\ y + 4z = 0 \\ z = r \end{array} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4r \\ -4r \\ r \end{pmatrix} = r\begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix} \qquad \vec{v}_3 = \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}$$

11. (6 pts) Find the eigenpolynomials for each eigenvalue. Call them q_1, q_2 and q_3.
Verify they are eigenpolynomials by checking that $L(q_k) = \lambda_k q_k$ using the definition of L.

$$q_1 = e\vec{v}_1 = 1 \qquad q_2 = e\vec{v}_2 = -2 + x \qquad q_3 = e\vec{v}_3 = 4 - 4x + x^2$$
$$L(q_1) = L(1) = (x-2)(0) = 0 = 0q_1$$
$$L(q_2) = L(-2+x) = (x-2)(1) = -2 + x = 1q_2$$
$$L(q_3) = L(4 - 4x + x^2) = (x-2)(-4+2x) = 8 - 8x + 2x^2 = 2(4 - 4x + x^2) = 2q_3$$

12. (6 pts) The eigenpolynomials $q = (q_1, q_2, q_3)$ form a second basis for P_3.
Find the matrix of the linear map L relative to the q basis. Call it $\underset{q\leftarrow q}{D}$.

$$L(q_1) = 0q_1 \qquad L(q_2) = 1q_2 \qquad L(q_3) = 2q_3 \qquad \underset{q\leftarrow q}{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

13. (15 pts) Find the change of basis matrices $\underset{e\leftarrow q}{C}$ and $\underset{q\leftarrow e}{C}$.

$$q_1 = 1 = e_1 \qquad q_2 = -2 + x = -2e_1 + e_2 \qquad q_3 = 4 - 4x + x^2 = 4e_1 - 4e_2 + e_3$$

$$\underset{e\leftarrow q}{C} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$ Equivalently, the eigenvectors go in the columns.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \qquad \underset{q\leftarrow e}{C} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

14. (5 pts) Verify your matrices satisfy $\underset{e\leftarrow q}{C}\ \underset{q\leftarrow q}{D}\ \underset{q\leftarrow e}{C} = \underset{e\leftarrow e}{A}$.

$$\underset{e\leftarrow q}{C}\ \underset{q\leftarrow q}{D}\ \underset{q\leftarrow e}{C} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -2 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 2 \end{pmatrix} = \underset{e\leftarrow e}{A}$$

15. (5 pts) Compute $\left(\underset{e\leftarrow e}{A}\right)^5$. $\qquad \left(\underset{e\leftarrow e}{A}\right)^5 = \left(\underset{e\leftarrow q}{C}\ \underset{q\leftarrow q}{D}\ \underset{q\leftarrow e}{C}\right)^5 = \underset{e\leftarrow q}{C}\left(\underset{q\leftarrow q}{D}\right)^5 \underset{q\leftarrow e}{C}$

$$= \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 32 \end{pmatrix}\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 32 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 120 \\ 0 & 1 & -124 \\ 0 & 0 & 32 \end{pmatrix}$$