1. Consider the vector space $V = \text{Span}(1, e^x, e^{-x})$ with the usual addition and scalar multiplication of functions.

a. Show $\vec{e}_1 = 1$, $\vec{e}_2 = e^x$ and $\vec{e}_3 = e^{-x}$ are a basis for V. What is the dimension of V?
HINT: Since they already span V, all you need to show is linear independence.

b. Show $\vec{E}_1 = 1$, $\vec{E}_2 = \sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\vec{E}_3 = \cosh x = \dfrac{e^x + e^{-x}}{2}$ are another basis for V.
HINT: Why do you only need to show one of spanning or linear independence?

c. Find $\underset{e\leftarrow E}{C}$, the change of basis matrix from the E-basis to the e-basis.
NOTE: If the bases are taken as rows:

$$e = (\vec{e}_1, \vec{e}_2, \vec{e}_3) = (1, e^x, e^{-x}) \quad \text{and} \quad E = \left(\vec{E}_1, \vec{E}_2, \vec{E}_3\right) = (1, \sinh x, \cosh x)$$

and the components of a vector $\vec{v}$ are columns $(\vec{v})_e$ and $(\vec{v})_E$ satisfying $\vec{v} = e(\vec{v})_e = E(\vec{v})_E$ then this matrix satisfies: $(\vec{v})_e = \underset{e\leftarrow E}{C}(\vec{v})_E$ and $E = e\underset{e\leftarrow E}{C}$.

d. Find $\underset{E\leftarrow e}{C}$, the change of basis matrix from the e-basis to the E-basis.

e. For the function $q = 7 + 4\sinh x - 2\cosh x$, find the components relative to the E-basis.
Then use $\underset{e\leftarrow E}{C}$ to find the components relative to the e-basis.
Then check your work by substituting $\sinh x$ and $\cosh x$ directly into the function.

f. For the function $r = 5 - 2e^x + 4e^{-x}$, find the components relative to the e-basis.
Then use $\underset{E\leftarrow e}{C}$ to find the components of r relative to the E-basis.
Then check your work by substituting $\sinh x$ and $\cosh x$ into the answer.

2. Consider the vector space $W = \text{Span}(e^{-x}, 1, e^x, e^{2x})$ with the usual addition and scalar multiplication of functions. Two bases are:

$$\vec{f}_1 = e^{-x}, \quad \vec{f}_2 = 1, \quad \vec{f}_3 = e^x, \quad \vec{f}_4 = e^{2x}$$

and

$$\vec{F}_1 = e^{-x}, \quad \vec{F}_2 = e^{-x} + 1, \quad \vec{F}_3 = 1 + e^x, \quad \vec{F}_4 = e^x + e^{2x}$$

a. Find $\underset{f\leftarrow F}{C}$, the change of basis matrix from the F-basis to the f-basis.

b. Find $\underset{F\leftarrow f}{C}$, the change of basis matrix from the f-basis to the F-basis.

3. With V and W as defined in #1 and #2, consider the function $L : W \to V$ given by

$$L(p) = \frac{dp}{dx} - 2p$$

a. First make sure the function L is well defined.
In other words, for $p = ae^{-x} + b + ce^{x} + de^{2x} \in W$, verify that $L(p) \in V$.

b. Show L is linear.

c. Find the $Ker(L)$. What is $\dim Ker(L)$?

d. Find the $Im(L)$. What is $\dim Im(L)$?

e. What Theorem relates $\dim Ker(L)$ and $\dim Im(L)$ to $\dim W$ or $\dim V$? Check it.

f. Let $q = 4e^{-x} - 2e^{x} + 2e^{2x}$ and compute $L(q)$ directly from the definition of L.

g. Find $\underset{e \leftarrow f}{A}$, the matrix of the linear map L from to the f-basis on W to the e-basis on V.

h. Compute $(q)_f$ and recompute $L(q) = L(4e^{-x} - 2e^{x} + 2e^{2x})$ using $\underset{e \leftarrow f}{A}$.

i. Find $\underset{E \leftarrow F}{B}$, the matrix of the linear map L from to the F-basis on W to the E-basis on V.
HINT: Use $\underset{e \leftarrow f}{A}$, and two of $\underset{e \leftarrow E}{C}$, $\underset{E \leftarrow e}{C}$, $\underset{f \leftarrow F}{C}$ and/or $\underset{F \leftarrow f}{C}$.

j. Use $\underset{F \leftarrow f}{C}$ to compute $(q)_F$, the components of $q = 4e^{-x} - 2e^{x} + 2e^{2x}$ relative to the F-basis.

k. Recompute $L(q) = L(4e^{-x} - 2e^{x} + 2e^{2x})$ using $(q)_F$ and $\underset{E \leftarrow F}{B}$.
Then check your work by substituting $\sinh x$ and $\cosh x$ into the answer.