1. (35 points+5 e.c.) Consider the vector space $M(2, 2)$ of $2 \times 2$ real matrices.
Consider the bases

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$F_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad F_4 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Consider the function $L : M(2, 2) \to M(2, 2)$ given by $L(X) = 2X - X^T$

a. (5 pts) Show $L$ is linear.

b. (5 pts) Find the matrix of $L$ relative to the $E$-basis. Call it $A_{E-E}$.

HINT: $A_{E-E}$ is NOT a $2 \times 2$ matrix!
c. (5 pts) Find the change of basis matrix $C_{E \rightarrow F}$ from the $F$-basis to the $E$-basis.

d. (5 pts) Find the change of basis matrix $C_{F \rightarrow E}$ from the $E$-basis to the $F$-basis.

e. (10 pts) Find the matrix of $L$ relative to the $F$-basis. Call it $B_{F \rightarrow F}$. 
f. (5 pts) Find $B_{F-F}$ by a second method.

g. (5 pts Extra Credit) Looking at your solutions to (b), (e) and (f), find 3 linearly independent eigenvectors of $L$. What are their eigenvalues?
2. (30 points) Consider the vector space $M(2,2)$ of $2 \times 2$ real matrices. Consider the function of two matrices

$$\langle X, Y \rangle = \text{tr}(X D Y^T)$$

where $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $\text{tr}$ means trace (sum of diagonal elements) and $^T$ means transpose.

a. (10 pts) Show $\langle X,Y \rangle$ is an inner product.

HINT: First compute the inner product of $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Y = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

b. (10 pts) Find the angle between the matrices $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

c. (10 pts) Let $V = \text{Span}\{P,Q\}$ where $P$ and $Q$ are given in (b).

- Find $V^\perp$ the orthogonal subspace to $V$ within $M(2,2)$.

- Find a basis for $V^\perp$ and its dimension.
3. (35 points + 5 e.c.) Consider the matrix \( A = \begin{pmatrix} 0 & 6 \\ -1 & 5 \end{pmatrix} \).

a. (15 pts) Find the eigenvalues and corresponding eigenvectors of \( A \).

b. (10 pts) Find the diagonalizing matrix \( X \) so that \( A = XDX^{-1} \) where \( D \) is diagonal. What are \( X^{-1} \) and \( D \)? Just state them. No derivation.

\[
X = \begin{pmatrix} \hspace{1cm} \\ \hspace{1cm} \end{pmatrix} \quad X^{-1} = \begin{pmatrix} \hspace{1cm} \\ \hspace{1cm} \end{pmatrix} \quad D = \begin{pmatrix} \hspace{1cm} \\ \hspace{1cm} \end{pmatrix}
\]

c. (5 pts) Find \( A^4 \).

d. (5 pts) Find \( \cos(\pi A) \).

e. (5 pts Extra Credit) What are the eigenvalues and corresponding eigenvectors of \( 5A \)?