Name: ______________________________

1	/20	4	/25
2	/20	5	/15
3	/20	Total	/100

MATH 311 Section 501 Spring 2013 P. Yasskin

Final

1. (20 points) Hams Duet is flying the Millenium Eagle through a region of intergalactic space containing a deadly hyperon vector field which is a function of position, $\vec{H} = (H_1(x,y,z), H_2(x,y,z), H_3(x,y,z))$. Of course, its magnitude is $M = \left|\vec{H}\right| = \sqrt{(H_1)^2 + (H_2)^2 + (H_3)^2}$. At stardate time $t = 21437.439$ years, Hams is located at the point $(x,y,z) = (5,-3,2)$ millilightyears and has velocity $\vec{v} = (3,-2,1)$ millilightyears/year. At that instant, he measures the hyperon density is

$$\vec{H} = (12\times 10^4, -3\times 10^4, 4\times 10^4) \quad \text{hyperons/millilightyear}^3$$

the gradients of its components are

$$\vec{\nabla}H_1 = (2,-1,3) \qquad \vec{\nabla}H_2 = (4,0,-1) \qquad \vec{\nabla}H_3 = (-2,1,3) \quad \text{hyperons/millilightyear}^4.$$

Find the **current hyperon magnitude** M and its **current rate of change** $\frac{dM}{dt}$ as seen by Hams?

HINT: Compute M and then the Jacobian matrices $\frac{D(M)}{D(H_1,H_2,H_3)}$, $\frac{D(H_1,H_2,H_3)}{D(x,y,z)}$ and $\frac{D(x,y,z)}{D(t)}$ and combine them to get $\frac{dM}{dt}$.

2. (20 points) Consider the vector space $(P_3)^2 = P_3 \times P_3$ of ordered pairs of polymonials of degree less than 3. For example, $(2+3x+4x^2, 5-4x-3x^2) \in (P_3)^2$. We take the "standard" basis to be:

$$\vec{e}_1 = (1,0), \quad \vec{e}_2 = (x,0), \quad \vec{e}_3 = (x^2,0), \quad \vec{e}_4 = (0,1), \quad \vec{e}_5 = (0,x), \quad \vec{e}_6 = (0,x^2)$$

and the alternate basis to be:

$$\vec{E}_1 = (1,0), \quad \vec{E}_2 = (1+x,0), \quad \vec{E}_3 = (1+x^2,0), \quad \vec{E}_4 = (0,1), \quad \vec{E}_5 = (0,1+x), \quad \vec{E}_6 = (0,1+x^2)$$

a. Find the change of basis matrices $\underset{E\leftarrow e}{C}$ and $\underset{e\leftarrow E}{C}$. Be sure to say which is which.

b. Find the components $(\vec{v})_e$ of the vector $\vec{v} = (2+3x+4x^2, 5-4x-3x^2)$ relative to the e-basis.

c. Use a change of basis matrix to find the components $(\vec{v})_E$ of the vector $\vec{v}$ relative to the E-basis.

d. Check your answer to (c) by hooking the components $(\vec{v})_E$ onto the E-basis vectors and simplifying.

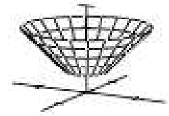
3. (20 points) Consider the subspace $V = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^4$ with the inner product $\langle \vec{p}, \vec{q} \rangle = \vec{p}^{\mathsf{T}} G \vec{q}$ where $^{\mathsf{T}}$ means transpose and

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}.$$

Find an orthonormal basis for V by applying the Gram-Schmidt procedure to the vectors $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

4. (25 points) Verify Stokes' Theorem $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial C} \vec{F} \cdot d\vec{s}$

for the slice of the cone C given by $z = \sqrt{x^2 + y^2}$ for $1 \le z \le 3$.



oriented down and out, and the vector field $\vec{F} = (-yz, xz, z^2)$.

Note: The boundary of the cone has 2 pieces:

the top circle, $x^2 + y^2 = 9$, and the bottom circle, $x^2 + y^2 = 1$.

Be sure to check the orientations. Use the following steps:

a. The cone may be parametrized as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

Compute the surface integral by successively finding:

$\vec{e}_r, \quad \vec{e}_\theta, \quad \vec{N}, \quad \vec{\nabla} \times \vec{F}, \quad \vec{\nabla} \times \vec{F}\Big|_{\vec{R}(r,\theta)}, \quad \iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$

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b. The top circle T may be parametrized as $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 3)$.
Compute the line integral over the top circle by successively finding:

$$\vec{v}, \quad \vec{F}\Big|_{\vec{r}(\theta)}, \quad \oint_T \vec{F} \cdot d\vec{s}$$

c. Compute the line integral over the bottom circle by successively finding:

$$\vec{r}(\theta), \quad \vec{v}, \quad \vec{F}\Big|_{\vec{r}(\theta)}, \quad \oint_B \vec{F} \cdot d\vec{s}$$

d. Combine the results from (b) and (c) to get $\oint_{\partial C} \vec{F} \cdot d\vec{s}$.

5. (15 points) Compute $\iint_H \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = (xy^2, yx^2, x^2 + y^2)$ over the hemisphere H given by $x^2 + y^2 + z^2 = 25$ with $z \geq 0$ oriented upward.

HINT: Use Gauss' Theorem to convert this surface integral into a volume integral over a solid hemisphere V and a surface integral over a disk D. Then add or subtract the answers to get the required integral. Be careful with the orientations.