1. (20 points) Hams Duet is flying the Millenium Eagle through a region of intergalactic space containing a deadly hyperon vector field which is a function of position,
\( \vec{H} = (H_1(x,y,z), H_2(x,y,z), H_3(x,y,z)) \). Of course, its magnitude is
\[ M = \sqrt{H_1^2 + H_2^2 + H_3^2} \] . At stardate time \( t = 21437.439 \) years, Hams is located at the point \( (x,y,z) = (5,-3,2) \) millilightyears and has velocity \( \vec{v} = (3,-2,1) \) millilightyears/year. At that instant, he measures the hyperon density is
\( \vec{H} = (12 \times 10^4, -3 \times 10^4, 4 \times 10^4) \) hyperons/millilightyear\(^3\)
the gradients of its components are
\[ \vec{v}H_1 = (2,-1,3) \quad \vec{v}H_2 = (4,0,-1) \quad \vec{v}H_3 = (-2,1,3) \] hyperons/millilightyear\(^4\).
Find the **current hyperon magnitude** \( M \) and its **current rate of change** \( \frac{dM}{dt} \) as seen by Hams?

**HINT:** Compute \( M \) and then the Jacobian matrices
\[ \frac{D(M)}{D(H_1, H_2, H_3)}, \quad \frac{D(H_1, H_2, H_3)}{D(x,y,z)} \] and combine them to get \( \frac{dM}{dt} \).
2. (20 points) Consider the vector space \((P_3)^2 = P_3 \times P_3\) of ordered pairs of polynomials of degree less than 3. For example, \((2 + 3x + 4x^2, 5 - 4x - 3x^2) \in (P_3)^2\). We take the "standard" basis to be:

\[
\vec{e}_1 = (1, 0), \quad \vec{e}_2 = (x, 0), \quad \vec{e}_3 = (x^2, 0), \quad \vec{e}_4 = (0, 1), \quad \vec{e}_5 = (0, x), \quad \vec{e}_6 = (0, x^2)
\]

and the alternate basis to be:

\[
\vec{E}_1 = (1, 0), \quad \vec{E}_2 = (1 + x, 0), \quad \vec{E}_3 = (1 + x^2, 0), \quad \vec{E}_4 = (0, 1), \quad \vec{E}_5 = (0, 1 + x), \quad \vec{E}_6 = (0, 1 + x^2)
\]

a. Find the change of basis matrices \(C_{E \leftrightarrow e}\) and \(C_{e \leftrightarrow E}\). Be sure to say which is which.

b. Find the components \((\vec{v})_e\) of the vector \(\vec{v} = (2 + 3x + 4x^2, 5 - 4x - 3x^2)\) relative to the \(e\)-basis.

c. Use a change of basis matrix to find the components \((\vec{v})_E\) of the vector \(\vec{v}\) relative to the \(E\)-basis.

d. Check your answer to (c) by hooking the components \((\vec{v})_E\) onto the \(E\)-basis vectors and simplifying.
3. (20 points) Consider the subspace $V = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^4$ with the inner product $\langle \vec{p}, \vec{q} \rangle = \vec{p}^\top G \vec{q}$ where $^\top$ means transpose and

$$
\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}.
$$

Find an orthonormal basis for $V$ by applying the Gram-Schmidt procedure to the vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$. 
4. (25 points) Verify Stokes’ Theorem \[ \oint_C \mathbf{\nabla} \times \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathcal{D}C} \mathbf{F} \cdot d\mathbf{S} \]
for the slice of the cone \( C \) given by \( z = \sqrt{x^2 + y^2} \) for \( 1 \leq z \leq 3 \) oriented down and out, and the vector field \( \mathbf{F} = (-yz, xz, z^2) \).

Note: The boundary of the cone has 2 pieces:
the top circle, \( x^2 + y^2 = 9 \), and the bottom circle, \( x^2 + y^2 = 1 \).

Be sure to check the orientations. Use the following steps:

a. The cone may be parametrized as \( \mathbf{R}(r, \theta) = (r \cos \theta, r \sin \theta, r) \)

Compute the surface integral by successively finding:
\[ \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{N}, \mathbf{\nabla} \times \mathbf{F}, \mathbf{\nabla} \times \mathbf{F} \mathbf{R}(r, \theta), \oint_C \mathbf{\nabla} \times \mathbf{F} \cdot d\mathbf{S} \]
b. The top circle $T$ may be parametrized as $\vec{r}(\theta) = (3 \cos \theta, 3 \sin \theta, 3)$. Compute the line integral over the top circle by successively finding:

$$\vec{v}, \quad \vec{F} \bigg|_{\vec{r}(\theta)}, \quad \oint_{T} \vec{F} \cdot d\vec{s}$$

c. Compute the line integral over the bottom circle by successively finding:

$$\vec{r}(\theta), \quad \vec{v}, \quad \vec{F} \bigg|_{\vec{r}(\theta)}, \quad \oint_{B} \vec{F} \cdot d\vec{s}$$

d. Combine the results from (b) and (c) to get $\oint_{C} \vec{F} \cdot d\vec{s}$. 

5. (15 points) Compute \( \int \int_H \vec{F} \cdot d\vec{S} \) for the vector field \( \vec{F} = (xy^2, yx^2, x^2 + y^2) \) over the hemisphere \( H \) given by \( x^2 + y^2 + z^2 = 25 \) with \( z \geq 0 \) oriented upward.

HINT: Use Gauss’ Theorem to convert this surface integral into a volume integral over a solid hemisphere \( V \) and a surface integral over a disk \( D \). Then add or subtract the answers to get the required integral. Be careful with the orientations.