1. Consider the vector space $V = \text{Span}(e^{-x}, 1, e^x, e^{2x})$ with the usual addition and scalar multiplication of functions. Two bases are:

$$\vec{f}_1 = e^{-x}, \quad \vec{f}_2 = 1, \quad \vec{f}_3 = e^x, \quad \vec{f}_4 = e^{2x}$$

and

$$\vec{F}_1 = e^{-x}, \quad \vec{F}_2 = e^{-x} + 1, \quad \vec{F}_3 = 1 + e^x, \quad \vec{F}_4 = e^x + e^{2x}$$

a. Find $\underset{f \leftarrow F}{C}$, the change of basis matrix from the F-basis to the f-basis.

b. Find $\underset{F \leftarrow f}{C}$, the change of basis matrix from the f-basis to the F-basis.

2. Consider the vector space $W = \text{Span}(1, e^x, e^{-x})$ with the usual addition and scalar multiplication of functions. Two bases are:

$$\vec{e}_1 = 1, \quad \vec{e}_2 = e^x, \quad \vec{e}_3 = e^{-x}$$

and

$$\vec{E}_1 = 1, \quad \vec{E}_2 = \sinh x = \frac{e^x - e^{-x}}{2} \quad \vec{E}_3 = \cosh x = \frac{e^x + e^{-x}}{2}$$

a. Find $\underset{e \leftarrow E}{C}$, the change of basis matrix from the E-basis to the e-basis.

b. Find $\underset{E \leftarrow e}{C}$, the change of basis matrix from the e-basis to the E-basis.

3. With V and W as defined in #1 and #2, consider the function $L : V \to W$ given by

$$L(p) = \frac{dp}{dx} - 2p$$

a. First make sure the function L is well defined.
In other words, for $p = ae^{-x} + b + ce^{x} + de^{2x} \in V$, verify that $L(p) \in W$.

b. Show L is linear.

c. Find the $Ker(L)$. Give a basis. What is $\dim Ker(L)$?
Remember: The basis vectors must be functions in V.

d. Find the $Im(L)$. Give a basis. What is $\dim Im(L)$?
Remember: The basis vectors must be functions in W.

e. Let $q = 4e^{-x} - 2e^{x} + 2e^{2x}$ and compute $L(q)$ directly from the definition of L.

f. Find $\underset{e\leftarrow f}{A}$, the matrix of the linear map L from to the f-basis on V to the e-basis on W.

g. Compute $(q)_f$ and recompute $L(q) = L(4e^{-x} - 2e^{x} + 2e^{2x})$ using $\underset{e\leftarrow f}{A}$.

h. Find $\underset{E\leftarrow F}{B}$, the matrix of the linear map L from to the F-basis on V to the E-basis on W.
HINT: Use $\underset{e\leftarrow f}{A}$, and two of $\underset{e\leftarrow E}{C}$, $\underset{E\leftarrow e}{C}$, $\underset{f\leftarrow F}{C}$ and/or $\underset{F\leftarrow f}{C}$.

i. Use $\underset{F\leftarrow f}{C}$ to compute $(q)_F$, the components of $q = 4e^{-x} - 2e^{x} + 2e^{2x}$ relative to the F-basis.

j. Recompute $L(q) = L(4e^{-x} - 2e^{x} + 2e^{2x})$ using $(q)_F$ and $\underset{E\leftarrow F}{B}$.
Then check your work by substituting $\sinh x$ and $\cosh x$ into the answer.