1. (20 points) Compute \( \iint_S \mathbf{\nabla} \times \mathbf{F} \cdot d\mathbf{S} \) for \( \mathbf{F} = (-y, x, z) \)
over the "clam shell" surface, \( S \), parametrized by
\[
\mathbf{R}(r, \theta) = (r \cos \theta, r \sin \theta, r \sin(6\theta))
\]
for \( r \leq 2 \) oriented upward.

HINTS: Use Stokes Theorem.

What is the value of \( r \) on the boundary?
2. (36 points) Let \( V = \text{Span}(e^{2x} + e^{-2x}, e^{2x} - e^{-2x}) \) be the vector space of functions spanned by the basis
\[
e_1 = e^{2x} + e^{-2x}, \quad e_2 = e^{2x} - e^{-2x}
\]
Consider the linear operator \( L : V \rightarrow V \) given by \( L(f) = 4 \frac{df}{dx} \). Our goals are to compute the eigenvalues and eigenfunctions of the linear operator \( L \), to find the similarity transformation which diagonalizes the matrix of \( L \) and use this similarity transformation to compute a matrix power.

a. (5 pts) Find the matrix of \( L \) relative to the \((e_1, e_2)\) basis. Call it \( A \).

b. (3 pts) Find the characteristic polynomial for \( A \).

Factor it and identify the eigenvalues of \( A \). These are also the eigenvalues of \( L \).

c. (8 pts) Find the eigenvector(s) of \( A \) for each eigenvalue, as vectors in \( \mathbb{R}^2 \).

Name them \( \vec{v}_1 \) and \( \vec{v}_2 \).

d. (6 pts) Convert the eigenvectors of \( A \) into eigenfunctions of \( L \) as functions in \( V \).

Name them \( f_1 \) and \( f_2 \) and simplify them.

Then compute \( L(f_1) \) and \( L(f_2) \) to verify \( f_1 \) and \( f_2 \) are eigenfunctions.

Hint: Remember that the components of \( \vec{v}_1 \) and \( \vec{v}_2 \) are components of \( f_1 \) and \( f_2 \) relative to the \((e_1, e_2)\) basis.
e. (3 pts) Using the eigenfunctions as a new \((f_1, f_2)\) basis for \(V\), find the matrix of \(L\) relative to the \((f_1, f_2)\) basis. Call it \(D_{f\to f}\).

f. (5 pts) Find the change of basis matrices \(C_{e\to f}\) and \(C_{f\to e}\) between the \((e_1, e_2)\) basis to the \((f_1, f_2)\) bases. Be sure to identify which is which.

g. (2 pts) \(A\) and \(D\) are related by a similarity transformation \(A = S^{-1} DS\). Identify \(S\) as \(C_{e\to f}\) or \(C_{f\to e}\).

h. (4 pts) Compute \(A^{10}\) and \(A^{15}\).
3. (30 points) The density, \( \rho \), of an ideal gas is related to its pressure, \( P \), and its absolute temperature, \( T \), by the equation \( \rho = \frac{P}{kT} \) where \( k \) is a constant which depends on the particular ideal gas. We are considering an ideal gas for which \( k = 10^{-4} \) atm\(\cdot\)m\(^3\)/kg/°K. At the current time, \( t = t_0 \), a flying robotic nanobot is located at \((x,y,z) = (2,1,3)^T \) m and has velocity \( \vec{v} = (.4,.5,.2)^T \) m/sec. The nanobot measures the current pressure is \( P = 2 \) atm while its gradient is \( \nabla P = (-.06,.02,.04) \) atm/m. Similarly, the nanobot measures the current temperature is \( T = 250 \) °K while its gradient is \( \nabla T = (3,-2,-4) \) °K/m.

a. (2 pts) Find the current density, \( \rho \).

b. (6 pts) Find the Jacobian matrix of the density \( \frac{D(\rho)}{D(P,T)} \) in general (in terms of symbols like \( \frac{\partial\rho}{\partial T} \)), then in terms of \( P \) and \( T \), and finally at the current time \( t = t_0 \).

c. (4 pts) Find the Jacobian matrix \( \frac{D(P,T)}{D(x,y,z)} \) in general (in terms of symbols like \( \frac{\partial P}{\partial y} \)) and then at the current time \( t = t_0 \).

d. (4 pts) Find the Jacobian matrix \( \frac{D(x,y,z)}{D(t)} \) in general and then at \( t = t_0 \).
e. (6 pts) Find the time rate of change of the pressure as seen by the nanobot, at the current time \( t = t_0 \). Is the pressure currently increasing or decreasing?

f. (8 pts) Find the time rate of change of the density as seen by the nanobot, at the current time \( t = t_0 \). Is the density currently increasing or decreasing?
4. (26 points) Compute the integral \( \iint x \, dA \) over the region in the first quadrant bounded by
\[ y = 1 + x^2, \quad y = 2 + x^2, \quad y = 3 - x^2, \quad \text{and} \quad y = 5 - x^2. \]

a. (4 pts) Define the curvilinear coordinates \( u \) and \( v \) by \( y = u + x^2 \) and \( y = v - x^2 \).
What are the 4 boundaries in terms of \( u \) and \( v \)?

b. (4 pts) Solve for \( x \) and \( y \) in terms of \( u \) and \( v \). Express the results as a position vector.

c. (4 pts) Find the coordinate tangent vectors:

d. (8 pts) Compute the Jacobian factor:

e. (6 pts) Compute the integral: