1. (20 points) Compute $\int_S \nabla \times \vec{F} \cdot d\vec{S}$ for $\vec{F} = (-y, x, z)$

over the "clam shell" surface, $S$, parametrized by

$$\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, r \sin(6\theta))$$

for $r \leq 2$ oriented upward.

HINTS: Use Stokes Theorem.

What is the value of $r$ on the boundary?

Stokes Theorem says

$$\oint_C \vec{F} \cdot d\vec{s} = \int_S \nabla \times \vec{F} \cdot d\vec{S}$$

where $\partial S$ is the boundary curve.

Since $r = 2$ on the boundary, the boundary curve is $\vec{r}(\theta) = \vec{R}(2, \theta) = (2 \cos \theta, 2 \sin \theta, 2 \sin(6\theta))$

The velocity is $\vec{v} = (-2 \sin \theta, 2 \cos \theta, 12 \cos(6\theta))$

The vector field on the boundary is $\vec{F}(\vec{r}(\theta)) = (-y, x, z) = (-2 \sin \theta, 2 \cos \theta, 2 \sin(6\theta))$

$$\vec{F} \cdot \vec{v} = 4 \sin^2 \theta + 4 \cos^2 \theta + 24 \sin(6\theta) \cos(6\theta) = 4 + 24 \sin(6\theta) \cos(6\theta)$$

$$\int \nabla \times \vec{F} \cdot d\vec{S} = \int \nabla \times \vec{F} \cdot d\vec{s} = \int_0^{2\pi} 2\vec{F} \cdot \vec{v} d\theta = \int_0^{2\pi} 4 + 24 \sin(6\theta) \cos(6\theta) d\theta = \left[ 4\theta + 2 \sin^2(6\theta) \right]_0^{2\pi} = 8\pi$$

This can also be done directly without using Stokes Theorem.
2. (36 points) Let $V = \text{Span}(e^{2x} + e^{-2x}, e^{2x} - e^{-2x})$ be the vector space of functions spanned by the basis

$$e_1 = e^{2x} + e^{-2x}, \quad e_2 = e^{2x} - e^{-2x}$$

Consider the linear operator $L : V \to V$ given by $L(f) = 4 \frac{df}{dx}$. Our goals are to compute the eigenvalues and eigenfunctions of the linear operator $L$, to find the similarity transformation which diagonalizes the matrix of $L$ and use this similarity transformation to compute a matrix power.

a. (5 pts) Find the matrix of $L$ relative to the $(e_1, e_2)$ basis. Call it $A$.

$$L(e_1) = L(e^{2x} + e^{-2x}) = 4 \frac{d(e^{2x} + e^{-2x})}{dx} = 8e^{2x} - 8e^{-2x} = 8e_2$$

$$L(e_2) = L(e^{2x} - e^{-2x}) = 4 \frac{d(e^{2x} - e^{-2x})}{dx} = 8e^{2x} + 8e^{-2x} = 8e_1$$

$$A = \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix}$$

b. (3 pts) Find the characteristic polynomial for $A$.

Factor it and identify the eigenvalues of $A$. These are also the eigenvalues of $L$.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 8 \\ 8 & -\lambda \end{vmatrix} = \lambda^2 - 64 = (\lambda + 8)(\lambda - 8) \quad \lambda = -8, 8$$

c. (8 pts) Find the eigenvector(s) of $A$ for each eigenvalue, as vectors in $\mathbb{R}^2$.

Name them $\vec{v}_1$ and $\vec{v}_2$.

$$\lambda = -8: \quad \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -r \\ r \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\lambda = 8: \quad \begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} r \\ r \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

d. (6 pts) Convert the eigenvectors of $A$ into eigenfunctions of $L$ as functions in $V$.

Name them $f_1$ and $f_2$ and simplify them.

Then compute $L(f_1)$ and $L(f_2)$ to verify $f_1$ and $f_2$ are eigenfunctions.

Hint: Remember that the components of $\vec{v}_1$ and $\vec{v}_2$ are components of $f_1$ and $f_2$ relative to the $(e_1, e_2)$ basis.

$$f_1 = -1e_1 + 1e_2 = -(e^{2x} + e^{-2x}) + (e^{2x} - e^{-2x}) = -2e^{-2x}$$

$$f_2 = 1e_1 + 1e_2 = (e^{2x} + e^{-2x}) + (e^{2x} - e^{-2x}) = 2e^{2x}$$

$$L(f_1) = 4 \frac{d(-2e^{-2x})}{dx} = 16e^{-2x} = -8(-2e^{-2x}) = -8f_1$$

$$L(f_2) = 4 \frac{d(2e^{2x})}{dx} = 16e^{2x} = 8(2e^{2x}) = 8f_2$$
e. (3 pts) Using the eigenfunctions as a new \((f_1, f_2)\) basis for \(V\), find the matrix of \(L\) relative to the \((f_1, f_2)\) basis. Call it \(D\).

Since \((f_1, f_2)\) is a basis of eigenvectors, the matrix of \(L\) relative to the \((f_1, f_2)\) basis will be diagonal and the diagonal entries will be the eigenvalues.

\[
D_{f \rightarrow f} = \begin{pmatrix} -8 & 0 \\ 0 & 8 \end{pmatrix}
\]

f. (5 pts) Find the change of basis matrices \(e_{e \rightarrow f}\) and \(e_{f \rightarrow e}\) between the \((e_1, e_2)\) basis to the \((f_1, f_2)\) bases. Be sure to identify which is which.

\[
f_1 = -1e_1 + 1e_2 \quad \quad \quad e_{e \rightarrow f} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}
\]

\[
f_2 = 1e_1 + 1e_2 \quad \quad \quad e_{f \rightarrow e} = \left( e_{e \rightarrow f} \right)^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}
\]

g. (2 pts) \(A\) and \(D\) are related by a similarity transformation \(A = S^{-1}DS\).
Identify \(S\) as \(e_{e \rightarrow f}\) or \(e_{f \rightarrow e}\).

Since \(A = C_{e \rightarrow f} D_{f \rightarrow e} C_{e \rightarrow f}\) we identify \(S = C_{e \rightarrow f}\).

h. (4 pts) Compute \(A^{10}\) and \(A^{15}\).

With \(D = \begin{pmatrix} -8 & 0 \\ 0 & 8 \end{pmatrix} = 8 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\) we have

\[
D^{10} = \begin{pmatrix} 8^{10} & 0 \\ 0 & 8^{10} \end{pmatrix} = 8^{10} \mathbf{1} \quad \text{and} \quad A^{10} = (S^{-1}DS)^{10} = S^{-1}D^{10}S = 8^{10}S^{-1}\mathbf{1}S = 8^{10}\mathbf{1} = \begin{pmatrix} 8^{10} & 0 \\ 0 & 8^{10} \end{pmatrix}
\]

\[
D^{15} = 8^{15} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A^{15} = (S^{-1}DS)^{15} = S^{-1}D^{15}S = 8^{15}S^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}S = 8^{14}S^{-1}DS = 8^{14}A = 8^{14} \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 8^{15} \\ 8^{15} & 0 \end{pmatrix}
\]
The density, $\rho$, of an ideal gas is related to its pressure, $P$, and its absolute temperature, $T$, by the equation $\rho = \frac{P}{kT}$, where $k$ is a constant which depends on the particular ideal gas. We are considering an ideal gas for which $k = 10^{-4}$ atm·m$^3$/kg·°K. At the current time, $t = t_0$, a flying robotic nanobot is located at $(x, y, z) = (2, 1, 3)^T$ m and has velocity $\vec{v} = (0.4, 0.5, 0.2)^T$ m/sec. The nanobot measures the current pressure is $P = 2$ atm while its gradient is $\nabla P = (-0.06, 0.02, 0.04)$ atm/m. Similarly, the nanobot measures the current temperature is $T = 250$ °K while its gradient is $\nabla T = (3, -2, -4)$ °K/m.

a. (2 pts) Find the current density, $\rho$.

$$\rho = \frac{P}{kT} = \frac{2 \text{ atm}}{\left(10^{-4} \text{ atm} \cdot \text{m}^3/\text{kg} \cdot \text{°K}\right) \left(250 \text{ °K}\right)} = 80 \text{ kg/m}^3$$

b. (6 pts) Find the Jacobian matrix of the density $\frac{D(\rho)}{D(P, T)}$ in general (in terms of symbols like $\frac{\partial \rho}{\partial T}$), then in terms of $P$ and $T$, and finally at the current time $t = t_0$.

$$\frac{D(\rho)}{D(P, T)} = \left(\frac{\partial \rho}{\partial P}, \frac{\partial \rho}{\partial T}\right) = \left(\frac{1}{kT^2}, -\frac{P}{kT^2}\right) \quad \frac{D(\rho)}{D(P, T)} \bigg|_{t=t_0} = \left(\frac{1}{10^{-4} \cdot 250}, -\frac{2}{10^{-4}(250)^2}\right) = (40, -0.32)$$

c. (4 pts) Find the Jacobian matrix $\frac{D(P, T)}{D(x, y, z)}$ in general (in terms of symbols like $\frac{\partial P}{\partial y}$) and then at the current time $t = t_0$.

$$\frac{D(P, T)}{D(x, y, z)} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{pmatrix} = \begin{pmatrix} \nabla P & \nabla T \end{pmatrix} \quad \frac{D(P, T)}{D(x, y, z)} \bigg|_{t=t_0} = \begin{pmatrix} -0.06 & 0.02 & 0.04 \\ 3 & -2 & -4 \end{pmatrix}$$

d. (4 pts) Find the Jacobian matrix $\frac{D(x, y, z)}{D(t)}$ in general and then at $t = t_0$.

$$\frac{D(x, y, z)}{D(t)} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \vec{v} \quad \frac{D(x, y, z)}{D(t)} \bigg|_{t=t_0} = \frac{\vec{v}(t_0)}{t_0} = \begin{pmatrix} 0.4 \\ 0.5 \\ 0.2 \end{pmatrix}$$
e. (6 pts) Find the time rate of change of the pressure as seen by the nanobot, at the current time \( t = t_0 \). Is the pressure currently increasing or decreasing?

\[
\frac{dP}{dt} \Big|_{t=t_0} = \vec{\nabla}P \big|_{t=t_0} \cdot \vec{v}(t_0) = (-.06, .02, .04) \cdot \begin{pmatrix} .4 \\ .5 \\ .2 \end{pmatrix} = -.024 + .01 + .008 = -.006
\]

The pressure is decreasing.

f. (8 pts) Find the time rate of change of the density as seen by the nanobot, at the current time \( t = t_0 \). Is the density currently increasing or decreasing?

\[
\frac{dp}{dt} \big|_{t=t_0} = \frac{D(\rho)}{D(P,T)} \big|_{t=t_0} \frac{D(P,T)}{D(x,y,z)} \big|_{t=t_0} = (40, -32) \begin{pmatrix} -.06 & .02 & .04 \\ 3 & -2 & -4 \end{pmatrix} \begin{pmatrix} .4 \\ .5 \\ .2 \end{pmatrix} = (40, -32) \begin{pmatrix} -.006 \\ -.6 \end{pmatrix} = -.24 + .192 = -0.048
\]

The density is decreasing.
4. (26 points) Compute the integral \( \iint x \, dA \) over the region in the first quadrant bounded by 
\[ y = 1 + x^2, \quad y = 2 + x^2, \quad y = 3 - x^2, \quad \text{and} \quad y = 5 - x^2. \]

a. (4 pts) Define the curvilinear coordinates \( u \) and \( v \) by \( y = u + x^2 \) and \( y = v - x^2 \).

What are the 4 boundaries in terms of \( u \) and \( v \)?
\[ u = 1 \quad u = 2 \quad v = 3 \quad v = 5 \]

b. (4 pts) Solve for \( x \) and \( y \) in terms of \( u \) and \( v \). Express the results as a position vector.

Add and subtract: 
\[ 2y = u + x^2 + v - x^2 = u + v \quad y = \frac{u + v}{2} \]
\[ y - y = u + x^2 - v + x^2 = u - v + 2x^2 \quad 2x^2 = v - u \quad x = \frac{\sqrt{v - u}}{2} \]

\( \vec{r}(u,v) = (x(u,v), y(u,v)) = \left( \frac{\sqrt{v - u}}{2}, \frac{u + v}{2} \right) \)

c. (4 pts) Find the coordinate tangent vectors:
\[ \vec{e}_u = \frac{\partial \vec{r}}{\partial u} = \left( \frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{v - u}}, \frac{1}{2} \right) \]
\[ \vec{e}_v = \frac{\partial \vec{r}}{\partial v} = \left( \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v - u}}, \frac{1}{2} \right) \]

d. (8 pts) Compute the Jacobian factor:

\[ \frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} \frac{1}{2\sqrt{2}} & \frac{-1}{\sqrt{v - u}} & \frac{1}{2} \\ \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{v - u}} & \frac{1}{2} \end{vmatrix} = \frac{1}{4\sqrt{2}} \frac{-1}{\sqrt{v - u}} - \frac{1}{4\sqrt{2}} \frac{1}{\sqrt{v - u}} = \frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{v - u}} \]

\[ J = \left| \frac{\partial (x,y)}{\partial (u,v)} \right| = \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v - u}} \]

e. (6 pts) Compute the integral:

\[ \iint x \, dA = \int_3^5 \int_1^2 \frac{\sqrt{v - u}}{2\sqrt{2}} \frac{1}{\sqrt{v - u}} \, du \, dv = \int_3^5 \int_1^2 \frac{1}{4} \, du \, dv = \frac{1}{4} (5 - 3)(2 - 1) = \frac{1}{2} \]