1. (20 points) Compute $\int_S \mathbf{v} \times \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F} = (-y, x, z)$
over the "clam shell" surface, $S$, parametrized by
$$\mathbf{R}(r, \theta) = (r \cos \theta, r \sin \theta, r \sin(5\theta))$$
for $r \leq 3$ oriented upward.

HINTS: Use Stokes Theorem.

What is the value of $r$ on the boundary?

Stokes Theorem says $\int_S \mathbf{v} \times \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$ where $\partial S$ is the boundary curve.

Since $r = 3$ on the boundary, the boundary curve is $\mathbf{R}(\theta) = \mathbf{R}(3, \theta) = (3 \cos \theta, 3 \sin \theta, 3 \sin(5\theta))$
The velocity is $\mathbf{v} = (-3 \sin \theta, 3 \cos \theta, 15 \sin(5\theta))$
The vector field on the boundary is $\mathbf{F}(\mathbf{R}(\theta)) = (-y, x, z) = (-3 \sin \theta, 3 \cos \theta, 3 \sin(5\theta))$
$\mathbf{F} \cdot \mathbf{v} = 9 \sin^2 \theta + 9 \cos^2 \theta + 45 \sin(5\theta) \cos(5\theta) = 9 + 45 \sin(5\theta) \cos(5\theta)$
$$\int \mathbf{v} \times \mathbf{F} \cdot d\mathbf{S} = \int \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F} \cdot \mathbf{v} d\theta = \int_0^{2\pi} 9 + 45 \sin(5\theta) \cos(5\theta) d\theta = \left[ 9\theta + \frac{9}{2} \sin^2 (5\theta) \right]_0^{2\pi} = 18\pi$$
This can also be done directly without using Stokes Theorem.
2. (36 points) Let \( V = \text{Span}(e^{2x} + e^{-2x}, e^{2x} - e^{-2x}) \) be the vector space of functions spanned by the basis 
\[ e_1 = e^{2x} + e^{-2x}, \quad e_2 = e^{2x} - e^{-2x} \]
Consider the linear operator \( L : V \rightarrow V \) given by \( L(f) = 3 \frac{df}{dx} \). Our goals are to compute the eigenvalues and eigenfunctions of the linear operator \( L \), to find the similarity transformation which diagonalizes the matrix of \( L \) and use this similarity transformation to compute a matrix power.

a. (5 pts) Find the matrix of \( L \) relative to the \((e_1, e_2)\) basis. Call it \( A \).
\[
L(e_1) = L(e^{2x} + e^{-2x}) = 3 \frac{d(e^{2x} + e^{-2x})}{dx} = 6e^{2x} - 6e^{-2x} = 6e_2
\]
\[
L(e_2) = L(e^{2x} - e^{-2x}) = 3 \frac{d(e^{2x} - e^{-2x})}{dx} = 6e^{2x} + 6e^{-2x} = 6e_1
\]
\[
A = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}
\]

b. (3 pts) Find the characteristic polynomial for \( A \).
Factor it and identify the eigenvalues of \( A \). These are also the eigenvalues of \( L \).
\[
\det(A - \lambda I) = \begin{vmatrix} -\lambda & 6 \\ 6 & -\lambda \end{vmatrix} = \lambda^2 - 36 = (\lambda + 6)(\lambda - 6) \quad \lambda = -6, 6
\]

c. (8 pts) Find the eigenvector(s) of \( A \) for each eigenvalue, as vectors in \( \mathbb{R}^2 \).
Name them \( \vec{v}_1 \) and \( \vec{v}_2 \).
\[
\lambda = -6: \quad \begin{pmatrix} 6 & 6 \\ 6 & 6 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -r \\ r \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}
\]
\[
\lambda = 6: \quad \begin{pmatrix} -6 & 6 \\ 6 & -6 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} r \\ r \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

d. (6 pts) Convert the eigenvectors of \( A \) into eigenfunctions of \( L \) as functions in \( V \).
Name them \( f_1 \) and \( f_2 \) and simplify them.
Then compute \( L(f_1) \) and \( L(f_2) \) to verify \( f_1 \) and \( f_2 \) are eigenfunctions.
Hint: Remember that the components of \( \vec{v}_1 \) and \( \vec{v}_2 \) are components of \( f_1 \) and \( f_2 \) relative to the \((e_1, e_2)\) basis.
\[
f_1 = -1e_1 + 1e_2 = -(e^{2x} + e^{-2x}) + (e^{2x} - e^{-2x}) = -2e^{-2x}
\]
\[
f_2 = 1e_1 + 1e_2 = (e^{2x} + e^{-2x}) + (e^{2x} - e^{-2x}) = 2e^{2x}
\]
\[
L(f_1) = 3 \frac{d(-2e^{-2x})}{dx} = 12e^{-2x} = -6(-2e^{-2x}) = -6f_1
\]
\[
L(f_2) = 3 \frac{d(2e^{2x})}{dx} = 12e^{2x} = 6(2e^{2x}) = 6f_2
\]
e. (3 pts) Using the eigenfunctions as a new \((f_1, f_2)\) basis for \(V\), find the matrix of \(L\) relative to the \((f_1, f_2)\) basis. Call it \(D_{f\to f}\).

Since \((f_1, f_2)\) is a basis of eigenvectors, the matrix of \(L\) relative to the \((f_1, f_2)\) basis will be diagonal and the diagonal entries will be the eigenvalues.

\[
D = \begin{pmatrix}
-6 & 0 \\
0 & 6
\end{pmatrix}
\]

f. (5 pts) Find the change of basis matrices \(C\) and \(C\) between the \((e_1, e_2)\) basis to the \((f_1, f_2)\) bases. Be sure to identify which is which.

\[
f_1 = -1e_1 + 1e_2 \\
f_2 = 1e_1 + 1e_2
\]

\[
C_{e\to f} = \begin{pmatrix}
-1 & 1 \\
1 & 1
\end{pmatrix} \quad C_{f\to e}^{-1} = \frac{1}{2} \begin{pmatrix}
1 & -1 \\
-1 & -1
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

\[
C_{e\to f} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
C_{f\to e} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
D_{e\to e} = \begin{pmatrix}
6^{12} & 0 \\
0 & 6^{12}
\end{pmatrix} \quad D_{f\to f} = \begin{pmatrix}
6^{12} & 0 \\
0 & 6^{12}
\end{pmatrix}
\]

\[
D_{e\to f} = \begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix} \quad D_{f\to e}^{-1} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
D_{e\to f} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
D_{f\to e}^{-1} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
A_{e\to e} = \begin{pmatrix}
6^{12} & 0 \\
0 & 6^{12}
\end{pmatrix} \quad A_{f\to f} = \begin{pmatrix}
6^{12} & 0 \\
0 & 6^{12}
\end{pmatrix}
\]

\[
A_{e\to f} = \begin{pmatrix}
6^{12} & 0 \\
0 & 6^{12}
\end{pmatrix} \quad A_{f\to e} = \begin{pmatrix}
6^{12} & 0 \\
0 & 6^{12}
\end{pmatrix}
\]

\[
A_{e\to e} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
A_{f\to f} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

g. (2 pts) \(A\) and \(D\) are related by a similarity transformation \(A = S^{-1}DS\). Identify \(S\) as \(C_{e\to f}\) or \(C_{f\to e}\).

Since \(A = C_{e\to e}D_{e\to f}C_{f\to e}\) we identify \(S = C_{f\to e}\).

h. (4 pts) Compute \(A_{12}\) and \(A_{25}\).

With \(D = \begin{pmatrix}
-6 & 0 \\
0 & 6
\end{pmatrix}\) we have

\[
D_{12} = \begin{pmatrix}
6^{12} & 0 \\
0 & 6^{12}
\end{pmatrix} = 6^{12}I \quad \text{and} \quad A_{12} = (S^{-1}DS)^{12} = S^{-1}D^{12}S = 6^{12}S^{-1}1S = 6^{12}I = \begin{pmatrix}
6^{12} & 0 \\
0 & 6^{12}
\end{pmatrix}
\]

\[
D_{25} = 6^{25}\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix} \quad \text{and} \quad A_{25} = (S^{-1}DS)^{25} = S^{-1}D^{25}S = 6^{25}S^{-1}\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}S
\]

\[
= 6^{24}S^{-1}DS = 6^{24}A = 6^{24}\begin{pmatrix}
0 & 6 \\
6 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 6^{25} \\
6^{25} & 0
\end{pmatrix}
\]
The density, $\rho$, of an ideal gas is related to its pressure, $P$, and its absolute temperature, $T$, by the equation $\rho = \frac{P}{kT}$, where $k$ is a constant which depends on the particular ideal gas. We are considering an ideal gas for which $k = 10^{-4}$ \text{ atm-m}^3/\text{kg-K}$. At the current time, $t = t_0$, a flying robotic nanobot is located at $(x, y, z) = (2, 1, 3)^T \text{ m}$ and has velocity $\vec{v} = (.4, .5, .2)^T \text{ m/sec}$. The nanobot measures the current pressure is $P = 2 \text{ atm}$ while its gradient is $\nabla P = (-.03, .01, .02) \text{ atm/m}$. Similarly, the nanobot measures the current temperature is $T = 250 \degree \text{ K}$ while its gradient is $\nabla T = (3, -2, -4) \degree \text{ K/m}$.

a. (2 pts) Find the current density, $\rho$.

$$\rho = \frac{P}{kT} = \frac{2 \text{ atm}}{(10^{-4} \text{ atm-m}^3/\text{kg-K}) (250 \degree \text{ K})} = 80 \text{ kg/m}^3$$

b. (6 pts) Find the Jacobian matrix of the density $\frac{D(\rho)}{D(P, T)}$ in general (in terms of symbols like $\frac{\partial \rho}{\partial T}$), then in terms of $P$ and $T$, and finally at the current time $t = t_0$.

$$\frac{D(\rho)}{D(P, T)} = \left( \frac{\partial \rho}{\partial P}, \frac{\partial \rho}{\partial T} \right) = \left( \frac{1}{kT}, \frac{-P}{kT^2} \right) \quad \left. \frac{D(\rho)}{D(P, T)} \right|_{t=t_0} = \left( \frac{1}{10^{-4} \cdot 250}, \frac{-2}{10^{-4}(250)^2} \right) = (40, -32)$$

c. (4 pts) Find the Jacobian matrix $\frac{D(P, T)}{D(x, y, z)}$ in general (in terms of symbols like $\frac{\partial P}{\partial y}$) and then at the current time $t = t_0$.

$$\frac{D(P, T)}{D(x, y, z)} = \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} & \frac{\partial P}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \\ \end{bmatrix} = \begin{bmatrix} \nabla P \\ \nabla T \end{bmatrix} \quad \left. \frac{D(P, T)}{D(x, y, z)} \right|_{t=t_0} = \begin{bmatrix} -0.03 & 0.01 & 0.02 \\ 3 & -2 & -4 \end{bmatrix}$$

d. (4 pts) Find the Jacobian matrix $\frac{D(x, y, z)}{D(t)}$ in general and then at $t = t_0$.

$$\frac{D(x, y, z)}{D(t)} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \vec{v} \quad \left. \frac{D(x, y, z)}{D(t)} \right|_{t=t_0} = \vec{v}(t_0) = \begin{bmatrix} .4 \\ .5 \\ .2 \end{bmatrix}$$
e. (6 pts) Find the time rate of change of the pressure as seen by the nanobot, at the current time $t = t_0$. Is the pressure currently increasing or decreasing?

$$\frac{dP}{dt} = \nabla P \bigg|_{t=t_0} \cdot \vec{v}(t_0) = (-.03, .01, .02) \cdot \begin{pmatrix} .4 \\ .5 \\ .2 \end{pmatrix} = -.012 + .005 + .004 = -.003$$

The pressure is decreasing.

f. (8 pts) Find the time rate of change of the density as seen by the nanobot, at the current time $t = t_0$. Is the density currently increasing or decreasing?

$$\frac{d\rho}{dt} = \frac{D(\rho)}{D(P, T)} \bigg|_{t=t_0} \frac{D(P, T)}{D(x, y, z)} \bigg|_{t=t_0} \frac{D(x, y, z)}{D(t)} \bigg|_{t=t_0}$$

$$= (40, -.32) \begin{pmatrix} -.03 & .01 & .02 \\ 3 & -2 & -4 \end{pmatrix} \begin{pmatrix} .4 \\ .5 \\ .2 \end{pmatrix} = (40, -.32) \begin{pmatrix} -.003 \\ -.6 \end{pmatrix} = -.12 + .192 = .072$$

The density is increasing.
4. (26 points) Compute the integral $\iint x\,dA$ over the region in the first quadrant bounded by $y = 1 + x^2$, $y = 3 + x^2$, $y = 4 - x^2$, and $y = 5 - x^2$.

a. (4 pts) Define the curvilinear coordinates $u$ and $v$ by $y = u + x^2$ and $y = v - x^2$. What are the 4 boundaries in terms of $u$ and $v$?

$u = 1 \quad u = 3 \quad v = 4 \quad v = 5$

b. (4 pts) Solve for $x$ and $y$ in terms of $u$ and $v$. Express the results as a position vector.

Add and subtract: $2y = u + x^2 + v - x^2 = u + v \quad y = \frac{u + v}{2}$

$y - y = u + x^2 - v + x^2 = u - v + 2x^2 \quad 2x^2 = v - u \quad x = \frac{\sqrt{v - u}}{\sqrt{2}}$

$r(u,v) = (x(u,v), y(u,v)) = \left( \frac{\sqrt{v - u}}{\sqrt{2}}, \frac{u + v}{2} \right)$

c. (4 pts) Find the coordinate tangent vectors:

$\vec{e}_u = \frac{\partial r}{\partial u} = \left( \frac{1}{2\sqrt{2}} \frac{-1}{\sqrt{v - u}}, \frac{1}{2} \right)$

$\vec{e}_v = \frac{\partial r}{\partial v} = \left( \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v - u}}, \frac{1}{2} \right)$

d. (8 pts) Compute the Jacobian factor:

$\frac{\partial (x,y)}{\partial (u,v)} = \left| \begin{array}{cc} \frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{v - u}} \\ \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{v - u}} \end{array} \right| = \frac{1}{4\sqrt{2}} - \frac{1}{\sqrt{v - u}} - \frac{1}{4\sqrt{2}} - \frac{1}{\sqrt{v - u}} = \frac{1}{2\sqrt{2}} - \frac{1}{\sqrt{v - u}}$

$J = \left| \frac{\partial (x,y)}{\partial (u,v)} \right| = \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v - u}}$

e. (6 pts) Compute the integral:

$\iint x\,dA = \int_4^5 \int_1^3 \frac{\sqrt{v - u}}{\sqrt{2}} \frac{1}{2\sqrt{2}} \frac{1}{\sqrt{v - u}} \, du \, dv = \int_4^5 \int_1^3 \frac{1}{4} \, du \, dv = \frac{1}{4} (5 - 4)(3 - 1) = \frac{1}{2}$