1. (20 points) Compute \( \iint_S \mathbf{V} \times \mathbf{F} \cdot d\mathbf{S} \) for \( \mathbf{F} = (-y, x, z) \) over the "clam shell" surface, \( S \), parametrized by

\[
\mathbf{R}(r, \theta) = (r \cos \theta, r \sin \theta, r \sin(6\theta))
\]

for \( r \leq 2 \) oriented upward.

HINTS: Use Stokes Theorem.
What is the value of \( r \) on the boundary?
2. (36 points) Let \( V = \text{Span}(e^{2x} + e^{-2x}, e^{2x} - e^{-2x}) \) be the vector space of functions spanned by the basis
\[
e_1 = e^{2x} + e^{-2x}, \quad e_2 = e^{2x} - e^{-2x}
\]
Consider the linear operator \( L : V \to V \) given by \( L(f) = 4 \frac{df}{dx} \). Our goals are to compute the eigenvalues and eigenfunctions of the linear operator \( L \), to find the similarity transformation which diagonalizes the matrix of \( L \) and use this similarity transformation to compute a matrix power.

a. (5 pts) Find the matrix of \( L \) relative to the \((e_1, e_2)\) basis. Call it \( A \).

b. (3 pts) Find the characteristic polynomial for \( A \).
Factor it and identify the eigenvalues of \( A \). These are also the eigenvalues of \( L \).

c. (8 pts) Find the eigenvector(s) of \( A \) for each eigenvalue, as vectors in \( \mathbb{R}^2 \).
Name them \( \vec{v}_1 \) and \( \vec{v}_2 \).

d. (6 pts) Convert the eigenvectors of \( A \) into eigenfunctions of \( L \) as functions in \( V \).
Name them \( f_1 \) and \( f_2 \) and simplify them.
Then compute \( L(f_1) \) and \( L(f_2) \) to verify \( f_1 \) and \( f_2 \) are eigenfunctions.
Hint: Remember that the components of \( \vec{v}_1 \) and \( \vec{v}_2 \) are components of \( f_1 \) and \( f_2 \) relative to the \((e_1, e_2)\) basis.
e. (3 pts) Using the eigenfunctions as a new \((f_1, f_2)\) basis for \(V\), find the matrix of \(L\) relative to the \((f_1, f_2)\) basis. Call it \(D_{f\rightarrow f}\).

f. (5 pts) Find the change of basis matrices \(C_{e\rightarrow f}\) and \(C_{f\rightarrow e}\) between the \((e_1, e_2)\) basis to the \((f_1, f_2)\) bases. Be sure to identify which is which.

g. (2 pts) \(A\) and \(D\) are related by a similarity transformation \(A = S^{-1}DS\). Identify \(S\) as \(C_{e\rightarrow f}\) or \(C_{f\rightarrow e}\).

h. (4 pts) Compute \(A^{10}\) and \(A^{15}\).
3. (30 points) The density, $\rho$, of an ideal gas is related to its pressure, $P$, and its absolute temperature, $T$, by the equation $\rho = \frac{P}{kT}$ where $k$ is a constant which depends on the particular ideal gas. We are considering an ideal gas for which $k = 10^{-4}$ atm·m$^3$/kg·°K. At the current time, $t = t_0$, a flying robotic nanobot is located at $(x, y, z) = (2, 1, 3)^T$ m and has velocity $\vec{v} = (4, 5, 2)^T$ m/sec. The nanobot measures the current pressure is $P = 2$ atm while its gradient is $\nabla P = (-0.06, 0.02, 0.04)$ atm/m. Similarly, the nanobot measures the current temperature is $T = 250$ °K while its gradient is $\nabla T = (3, -2, -4)$ °K/m.

a. (2 pts) Find the current density, $\rho$.

b. (6 pts) Find the Jacobian matrix of the density $\frac{D(\rho)}{D(P, T)}$ in general (in terms of symbols like $\frac{\partial \rho}{\partial T}$), then in terms of $P$ and $T$, and finally at the current time $t = t_0$.

c. (4 pts) Find the Jacobian matrix $\frac{D(P, T)}{D(x, y, z)}$ in general (in terms of symbols like $\frac{\partial P}{\partial y}$) and then at the current time $t = t_0$.

d. (4 pts) Find the Jacobian matrix $\frac{D(x, y, z)}{D(t)}$ in general and then at $t = t_0$. 
e. (6 pts) Find the time rate of change of the pressure as seen by the nanobot, at the current time $t = t_0$. Is the pressure currently increasing or decreasing?

f. (8 pts) Find the time rate of change of the density as seen by the nanobot, at the current time $t = t_0$. Is the density currently increasing or decreasing?
4. (26 points) Compute the integral $\iint x \, dA$ over the region in the first quadrant bounded by

\[ y = 1 + x^2, \quad y = 2 + x^2, \quad y = 3 - x^2, \quad \text{and} \quad y = 5 - x^2. \]

a. (4 pts) Define the curvilinear coordinates $u$ and $v$ by $y = u + x^2$ and $y = v - x^2$. What are the 4 boundaries in terms of $u$ and $v$?

b. (4 pts) Solve for $x$ and $y$ in terms of $u$ and $v$. Express the results as a position vector.

c. (4 pts) Find the coordinate tangent vectors:

d. (8 pts) Compute the Jacobian factor:

e. (6 pts) Compute the integral: