1. Hydrochloric acid \((HCl)\) and sodium hydroxide \((NaOH)\) react to produce sodium chloride \((NaCl)\) and water \((H_2O)\) according to the chemical equation:

\[
aHCl + bNaOH \rightarrow cNaCl + dH_2O
\]

Which of the following is the augmented matrix which is used to solve this chemical equation? (Put the elements in the order \(H\), \(Cl\), \(Na\), \(O\).)

\[
a. \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & -1 & 1 & 0 & 0 \\
-2 & 0 & 0 & -1 & 0
\end{pmatrix}
\]

\[
b. \begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 & 0
\end{pmatrix}
\]

\[
c. \begin{pmatrix}
1 & 1 & 0 & -2 & 0 \\
1 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0
\end{pmatrix}
\]

\[
d. \begin{pmatrix}
1 & 1 & 0 & 2 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0
\end{pmatrix}
\]

2. Suppose \(A\) is nilpotent, i.e., \(A^2 = 0\). Which of the following is true?

a. \(A + 1\) is invertible and \((A + 1)^{-1} = A - 1\)

b. \(A + 1\) is invertible and \((A + 1)^{-1} = 1 - A\)

c. \(A + 1\) is invertible and \((A + 1)^{-1} = 1 - 2A\)

d. \(A + 1\) is invertible and \((A + 1)^{-1} = 1 + A\)

e. \(A + 1\) is not invertible
3. Consider the vector space $M(2, 2)$ of $2 \times 2$ matrices with the basis 

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Which of the following are the components of the matrix $A = \begin{pmatrix} 9 & 5 \\ 1 & 1 \end{pmatrix}$ relative to the $E$ basis?

- a. $(A)_E = \begin{pmatrix} 2 & 1 & 4 & 3 \end{pmatrix}^T$
- b. $(A)_E = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T$
- c. $(A)_E = \begin{pmatrix} 4 & 5 & 2 & 3 \end{pmatrix}^T$
- d. $(A)_E = \begin{pmatrix} 5 & 4 & 3 & 2 \end{pmatrix}^T$
- e. $(A)_E = \begin{pmatrix} 5 & -4 & 3 & -2 \end{pmatrix}^T$

4. Consider the vector space $C_2([0, 1])$ of real valued function on the interval $[0, 1]$ whose second derivatives exist and are continuous with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, dx$. Consider the subspace $V = \text{Span}(x, x^2)$ spanned by the basis $v_1 = x$, $v_2 = x^2$. Which of the following is an orthonormal basis for $V$?

- a. $u_1 = x$ $u_2 = x^2$
- b. $u_1 = x$ $u_2 = x^2 - \frac{3}{4} x$
- c. $u_1 = \frac{3}{2} x$ $u_2 = \frac{5}{2} x^2$
- d. $u_1 = \frac{5}{2} x$ $u_2 = \frac{7}{2} x^2$
- e. $u_1 = \sqrt{2} x$ $u_2 = 4 \sqrt{5} x^2 - 3 \sqrt{5} x$
5. Consider the second derivative linear operator \( L : P_6 \to P_6 : L(p) = \frac{d^2 p}{dx^2} \) on the space of polynomials of degree less than \( 6 \). Find the kernel, \( \text{Ker}(L) \).

HINT: Let \( p = a + bx + cx^2 + dx^3 + ex^4 + fx^5 \).

a. \( \text{Ker}(L) = \text{Span}(1) \)
b. \( \text{Ker}(L) = \text{Span}(1, x) \)
c. \( \text{Ker}(L) = \text{Span}(1, x, x^2) \)
d. \( \text{Ker}(L) = \text{Span}(x, x^3, x^4, x^5) \)
e. \( \text{Ker}(L) = \text{Span}(x, x^2, x^3, x^4, x^5) \)

6. Consider the second derivative linear operator \( L : P_6 \to P_6 : L(p) = \frac{d^2 p}{dx^2} \) on the space of polynomials of degree less than \( 6 \). Find the image, \( \text{Im}(L) \).

HINT: Let \( p = a + bx + cx^2 + dx^3 + ex^4 + fx^5 \).

a. \( \text{Im}(L) = \text{Span}(1) \)
b. \( \text{Im}(L) = \text{Span}(1, x) \)
c. \( \text{Im}(L) = \text{Span}(1, x, x^2, x^3) \)
d. \( \text{Im}(L) = \text{Span}(x, x^2, x^3, x^4, x^5) \)
e. \( \text{Im}(L) = \text{Span}(x, x^2, x^3, x^4, x^5) \)

7. Compute the line integral \( \oint \vec{F} \cdot d\vec{s} \) clockwise around the complete boundary of the plus sign, shown at the right, for the vector field \( \vec{F} = (4x^3 + 2y, 4y^3 - 3x) \).

a. 25  
b. 5  
c. 0  
d. -5  
e. -25
Consider the vector space $M(2, 2)$ of $2 \times 2$ matrices. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Consider the linear function, $L : M(2, 2) \to M(2, 2) : L(X) = AX -XA$. Which of the following is not an eigenvalue and corresponding eigenmatrix (eigenvector) of $L$?

HINT: Let $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.

a. $\lambda = -1 \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

b. $\lambda = 0 \quad X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

c. $\lambda = 0 \quad X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

d. $\lambda = 1 \quad X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

e. $\lambda = 2 \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
9. (16 points) Which of the following is an inner product on \( \mathbb{R}^2 \)? If not, why not? Put \( \times \)'s in the correct boxes. No part credit.

Let \( \vec{x} = (x_1, x_2), \ \vec{y} = (y_1, y_2) \).

<table>
<thead>
<tr>
<th>[ (\vec{x}, \vec{y}) = ]</th>
<th>Inner Product?</th>
<th>Not</th>
<th>Not</th>
<th>Not</th>
<th>Positive but Not</th>
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</thead>
<tbody>
<tr>
<td>a. ( x_1y_1 + 2x_2y_2 )</td>
<td>Yes</td>
<td>No</td>
<td>Symmetric</td>
<td>Linear</td>
<td>Positive</td>
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<td>b. ( x_1^2y_1^2 + 2x_1^2y_2^2 )</td>
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<td>c. ( x_1y_1 + 2x_1y_2 + 2x_2y_2 )</td>
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<td>d. ( x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 )</td>
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<td>e. ( x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2 )</td>
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<td>f. ( x_1y_1 - x_1y_2 + x_2y_1 + 2x_2y_2 )</td>
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<tr>
<td>g. ( x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 )</td>
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<td>h. ( x_1y_1 - x_2y_2 )</td>
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</table>
10. (20 points) Let \( M(2,3) \) be the vector space of \( 2 \times 3 \) matrices. Consider the subspace \( V = \text{Span}(A_1, A_2, A_3, A_4) \) where

\[
A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 2 & 0 \\ 4 & 6 & -4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 0 & 2 \\ -3 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 9 & -6 \end{pmatrix}
\]

Find a basis for \( V \). What is the \( \text{dim} V \)?
11. (28 points) Compute \( \oiint_{H} \vec{F} \cdot d\vec{S} \) over the hemisphere \( z = \sqrt{25 - x^2 - y^2} \) oriented upward, for the vector field \( \vec{F} = (x^3 + 4y^2 + 4z^2, 4x^2 + y^3 + 4z^2, 4x^2 + 4y^2 + z^3) \).

HINT: Use Gauss' Theorem by following these steps:

a. Write out Gauss' Theorem for the Volume, \( V \), which is the solid hemisphere \( 0 \leq z \leq \sqrt{25 - x^2 - y^2} \). Split up the boundary, \( \partial V \), into two pieces, the hemisphere, \( H \), and the disk, \( D \), at the bottom. State orientations. Solve for the integral you want.

b. Compute the volume integral using spherical coordinates.

(continued)
c. Compute the other surface integral over $D$ by parametrizing the disk, computing the tangent vectors and normal vector, checking the orientation, evaluating the vector field on the disk and doing the integral.

d. Solve for the original integral.