1. Hydrochloric acid (HCl) and sodium hydroxide (NaOH) react to produce sodium chloride (NaCl) and water (H₂O) according to the chemical equation:

\[ aHCl + bNaOH \rightarrow cNaCl + dH₂O \]

Which of the following is the augmented matrix which is used to solve this chemical equation? 
(Put the elements in the order H, Cl, Na, O.)

a. \[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

b. \[
\begin{pmatrix}
1 & 1 & 0 & 2 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
\end{pmatrix}
\]

c. \[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

d. \[
\begin{pmatrix}
1 & 1 & 0 & -2 & 0 \\
1 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
\end{pmatrix}
\]

2. Suppose \( A \) is nilpotent, i.e. \( A^2 = 0 \). Which of the following is true?

a. \( A + 1 \) is invertible and \( (A + 1)^{-1} = A - 1 \)

b. \( A + 1 \) is invertible and \( (A + 1)^{-1} = 1 - 2A \)

c. \( A + 1 \) is invertible and \( (A + 1)^{-1} = 1 - A \)

d. \( A + 1 \) is invertible and \( (A + 1)^{-1} = 1 + A \)

e. \( A + 1 \) is not invertible
3. Consider the vector space \( M(2, 2) \) of \( 2 \times 2 \) matrices with the basis
\[
E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
Which of the following are the components of the matrix \( A = \begin{pmatrix} 9 & 5 \\ 1 & 1 \end{pmatrix} \) relative to the \( E \) basis?

a. \( (A)_E = \begin{pmatrix} 2 & 1 & 4 & 3 \end{pmatrix}^T \)
b. \( (A)_E = \begin{pmatrix} 5 & 4 & 3 & 2 \end{pmatrix}^T \)
c. \( (A)_E = \begin{pmatrix} 4 & 5 & 2 & 3 \end{pmatrix}^T \)
d. \( (A)_E = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T \)
e. \( (A)_E = \begin{pmatrix} 5 & -4 & 3 & -2 \end{pmatrix}^T \)

4. Consider the vector space \( C[0, 1] \) of real valued function on the interval \([0, 1]\) whose second derivatives exist and are continuous with the inner product
\[
\langle f, g \rangle = \int_0^1 f(x) g(x) \, dx.
\]
Consider the subspace \( V = \text{Span}(x, x^2) \) spanned by the basis \( v_1 = x, \ v_2 = x^2 \). Which of the following is an orthonormal basis for \( V \)?

a. \( u_1 = x \quad u_2 = x^2 \)
b. \( u_1 = \sqrt{\frac{3}{2}} x \quad u_2 = \sqrt{\frac{7}{2}} x^2 \)
c. \( u_1 = x \quad u_2 = x^2 - \frac{3}{4} x \)
d. \( u_1 = \sqrt{\frac{3}{2}} x \quad u_2 = \sqrt{\frac{5}{2}} x^2 \)
e. \( u_1 = \sqrt{2} x \quad u_2 = 4 \sqrt{5} x^2 - 3 \sqrt{5} x \)
5. Consider the second derivative linear operator \( L : P_6 \rightarrow P_6 : L(p) = \frac{d^2p}{dx^2} \) on the space of polynomials of degree less than 6. Find the image, \( \text{Im}(L) \).

HINT: Let \( p = a + bx + cx^2 + dx^3 + ex^4 + fx^5 \).

a. \( \text{Im}(L) = \text{Span}(1) \)
b. \( \text{Im}(L) = \text{Span}(1,x) \)
c. \( \text{Im}(L) = \text{Span}(1,x,x^2,x^3) \)
d. \( \text{Im}(L) = \text{Span}(x,x^2,x^3,x^4,x^5) \)
e. \( \text{Im}(L) = \text{Span}(x,x^2,x^3,x^4,x^5) \)

6. Consider the second derivative linear operator \( L : P_6 \rightarrow P_6 : L(p) = \frac{d^2p}{dx^2} \) on the space of polynomials of degree less than 6. Find the kernel, \( \text{Ker}(L) \).

HINT: Let \( p = a + bx + cx^2 + dx^3 + ex^4 + fx^5 \).

a. \( \text{Ker}(L) = \text{Span}(1) \)
b. \( \text{Ker}(L) = \text{Span}(1,x) \)
c. \( \text{Ker}(L) = \text{Span}(1,x,x^2) \)
d. \( \text{Ker}(L) = \text{Span}(x,x^2,x^3,x^4,x^5) \)
e. \( \text{Ker}(L) = \text{Span}(x,x^2,x^3,x^4,x^5) \)

7. Compute the line integral \( \int \vec{F} \cdot d\vec{s} \) clockwise around the complete boundary of the plus sign, shown at the right, for the vector field \( \vec{F} = (4x^3 + 2y, 4y^3 - 3x) \).

a. \(-25\)
b. \(-5\)
c. \(0\)
d. \(5\)
e. \(25\)
8. Consider the vector space $M(2, 2)$ of $2 \times 2$ matrices. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Consider the linear function, $L : M(2, 2) \rightarrow M(2, 2) : L(X) = AX -XA$. Which of the following is not an eigenvalue and corresponding eigenmatrix (eigenvector) of $L$?

HINT: Let $X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$.

\begin{align*}
\text{a.} & \quad \lambda = 2 \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\text{b.} & \quad \lambda = 1 \quad X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
\text{c.} & \quad \lambda = 0 \quad X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\
\text{d.} & \quad \lambda = 0 \quad X = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
\text{e.} & \quad \lambda = -1 \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{align*}
9. (16 points) Which of the following is an inner product on \( \mathbb{R}^2 \)? If not, why not?

Put 's in the correct boxes. No part credit.

Let \( \vec{x} = (x_1, x_2) \), \( \vec{y} = (y_1, y_2) \).

<table>
<thead>
<tr>
<th>(\vec{x}, \vec{y}) =</th>
<th>Inner Product?</th>
<th>Not</th>
<th>Not</th>
<th>Not</th>
<th>Positive but Not</th>
<th>Why not?</th>
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</thead>
<tbody>
<tr>
<td>a. ( x_1 y_1 + 2x_2 y_2 )</td>
<td>Yes</td>
<td>No</td>
<td>Symmetric</td>
<td>Linear</td>
<td>Positive</td>
<td>Positive Definite</td>
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<td>b. ( x_1 y_1 + 2x_1 y_2 + 2x_2 y_2 )</td>
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<td>c. ( x_1^2 y_1^2 + 2x_2^2 y_2^2 )</td>
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<td>d. ( x_1 y_1 + x_1 y_2 + x_2 y_1 + 2x_2 y_2 )</td>
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<td>e. ( x_1 y_1 - x_1 y_2 - x_2 y_1 + 2x_2 y_2 )</td>
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<td>f. ( x_1 y_1 - x_1 y_2 - x_2 y_1 + x_2 y_2 )</td>
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<td>g. ( x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2 )</td>
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<td>h. ( x_1 y_1 - x_2 y_2 )</td>
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10. (20 points) Let $M(2,3)$ be the vector space of $2 \times 3$ matrices.
Consider the subspace $V = \text{Span}(A_1, A_2, A_3, A_4)$ where

$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2 & 2 & 0 \\ 4 & 6 & -4 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 0 & 2 \\ -3 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -1 & 2 & 6 \\ -5 & 6 & -4 \end{pmatrix}$

Find a basis for $V$. What is the $\text{dim } V$?
11. (28 points) Compute \( \iint_H \vec{F} \cdot d\vec{S} \) over the hemisphere \( z = \sqrt{25-x^2-y^2} \) oriented upward, for the vector field \( \vec{F} = (x^3 - 4y^2 - 4z^2, -4x^2 + y^3 - 4z^2, -4x^2 - 4y^2 + z^3) \).

HINT: Use Gauss' Theorem by following these steps:

a. Write out Gauss' Theorem for the Volume, \( \text{V} \), which is the solid hemisphere \( 0 \leq z \leq \sqrt{25-x^2-y^2} \). Split up the boundary, \( \partial \text{V} \), into two pieces, the hemisphere, \( H \), and the disk, \( D \), at the bottom. State orientations. Solve for the integral you want.

b. Compute the volume integral using spherical coordinates.
c. Compute the other surface integral over $D$ by parametrizing the disk, computing the tangent vectors and normal vector, checking the orientation, evaluating the vector field on the disk and doing the integral.

d. Solve for the original integral.