Name______________________ ID__________

MATH 311 Exam 1 Spring 2001
Section 200 Solutions P. Yasskin

1	/15	5	/10
2	/30	6	/10
3	/ 5	7	/10
4	/10	8	/10

1. (15 points) Consider the three points

$$P = (1,2,3) \qquad Q = (1,3,4) \qquad R = (2,3,3)$$

a. (5 pts) Find parametric equations of the plane containing P, Q and R.

$$\overrightarrow{PQ} = Q - P = (0,1,1) \qquad \overrightarrow{PR} = R - P = (1,1,0) \qquad X = P + s\overrightarrow{PQ} + t\overrightarrow{PR}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+t \\ 2+s+t \\ 3+s \end{pmatrix}$$

b. (5 pts) Find a non-parametric equation of the plane containing P, Q and R.

$$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \vec{i}(-1) - \vec{j}(-1) + \vec{k}(-1) = (-1,1,-1)$$

$$\vec{N} \bullet X = \vec{N} \bullet P \qquad -x + y - z = -1 + 2 - 3 = -2$$

c. (5 pts) Find an equation of the line through P perpendicular to the plane containing P, Q and R.

$$X = P + t\vec{N} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1-t \\ 2+t \\ 3-t \end{pmatrix}$$

2. (30 points) Consider the equations

$$\begin{aligned} 2x + 4y + 6z &= 4 \\ v + y &= 2 \\ w + z &= b \\ 2v + w + x + 4y + 4z &= 3 \end{aligned}$$

a. (5 pts) Write out the augmented matrix for this system.

$$\left(\begin{array}{ccccc|c} 0 & 0 & 2 & 4 & 6 & 4 \\ 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & b \\ 2 & 1 & 1 & 4 & 4 & 3 \end{array}\right)$$

b. (10 pts) For what value(s) of b do there exist solutions?

$\frac{1}{2}R_1$ and reorder rows :
$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & b \\ 0 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 1 & 4 & 4 & 3 \end{array}\right)$$

$R_4 - 2R_1 - R_2 - R_3 \to R_4$:
$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & b \\ 0 & 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & -3-b \end{array}\right)$$

So there are solutions only if $b = -3$.

c. (10 pts) For those value(s) of b, find all solutions.

For $b = -3$, the equations reduce to

$$\begin{aligned} v + y &= 2 \\ w + z &= -3 \\ x + 2y + 3z &= 2 \end{aligned}$$

So y and z are arbitrary and the solution is

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2-s \\ -3-t \\ 2-2s-3t \\ s \\ t \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 2 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

d. (5 pts) Circle the geometric description of the solution set:

point, line, [2-plane], 3-plane, 4-plane, $\mathbf{R}^5$

3. (5 points) In $\mathbf{R}^5$ with coordinates (v,w,x,y,z), write out an equation of the 3-plane through the point $P = (5,4,3,2,1)$ with tangent vectors

$$\vec{a} = (2,1,3,0,4) \qquad \vec{b} = (1,-1,2,-2,3) \qquad \vec{c} = (2,1,3,-1,0)$$

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} + r\begin{pmatrix} 2 \\ 1 \\ 3 \\ 0 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 3 \\ -1 \\ 0 \end{pmatrix}$$

4. (10 points) Duke Skywater is flying the Millennium Eagle along the curve

$$\vec{r}(t) = (2\cos t, 3\sin t, t)$$

At $t = \frac{\pi}{2}$, he releases a garbage pod which travels along his tangent line with constant velocity (equal to his velocity at the time of release). Where is the garbage pod at $t = \pi$?

$$\vec{r}(t) = (2\cos t, 3\sin t, t) \qquad \vec{r}\left(\tfrac{\pi}{2}\right) = \left(2\cos\tfrac{\pi}{2}, 3\sin\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) = \left(0,3,\tfrac{\pi}{2}\right)$$

$$\vec{v}(t) = (-2\sin t, 3\cos t, 1) \qquad \vec{v}\left(\tfrac{\pi}{2}\right) = \left(-2\sin\tfrac{\pi}{2}, 3\cos\tfrac{\pi}{2}, 1\right) = (-2,0,1)$$

The tangent line is

$$(x,y,z) = \vec{r}_{\tan}(t) = \vec{r}\left(\tfrac{\pi}{2}\right) + \vec{v}\left(\tfrac{\pi}{2}\right)\left(t - \tfrac{\pi}{2}\right) = \left(0,3,\tfrac{\pi}{2}\right) + \left(t - \tfrac{\pi}{2}\right)(-2,0,1)$$

At $t = \pi$, the garbage pod is at

$$(x,y,z) = \vec{r}_{\tan}(\pi)\left(0,3,\tfrac{\pi}{2}\right) + \left(\tfrac{\pi}{2}\right)(-2,0,1) = (-\pi,3,\pi)$$

5. (10 points) Duke Skywater is flying the Millennium Eagle through the galactic polaron field. At $t = 20$, Duke's position is $\vec{r} = (20, 10, 30)$ lightyears and his velocity is $\vec{v} = (.1, .3, .2)\ \frac{\text{lightyears}}{\text{year}}$. At that time, he measures the density of polarons to be $\rho = 15 \times 10^6\ \frac{\text{polarons}}{\text{lightyear}^3}$ and the gradient of this density to be $\vec{\nabla}\rho = (2 \times 10^6, -1 \times 10^6, 3 \times 10^6)\ \frac{\text{polarons}}{\text{lightyear}^4}$.

a. What does he measure as the time rate of change the polaron density, $\frac{d\rho}{dt}$?

$$\frac{d\rho}{dt} = \vec{\nabla}\rho \cdot \vec{v} = (2 \times 10^6, -1 \times 10^6, 3 \times 10^6) \cdot (.1, .3, .2) = (.2 - .3 + .6) \times 10^6 = .5 \times 10^6 = 5 \times 10^5$$

b. Using a linear approximation, what would he expect the polaron density to be at the point $\vec{x} = (21, 12, 29)$?

$$\rho_{\tan}(\vec{x}) = \rho(\vec{r}) + \vec{\nabla}\rho \cdot (\vec{x} - \vec{r})$$
$$\rho_{\tan}(21, 12, 29) = 15 \times 10^6 + (2 \times 10^6, -1 \times 10^6, 3 \times 10^6) \cdot (1, 2, -1)$$
$$= 15 \times 10^6 + (2 - 2 - 3) \times 10^6 = 12 \times 10^6$$

6. (10 points) Duke Skywater is flying the Millennium Eagle through the galactic hyperon field. At $t = 20$, Duke's position is $\vec{r} = (20, 10, 30)$ lightyears and his velocity is $\vec{v} = (.1, .3, .2)\ \frac{\text{lightyears}}{\text{year}}$. At that time, he measures the hyperon field and its Jacobian to be

$$\vec{H} = \begin{pmatrix} 200 \\ 150 \\ 300 \end{pmatrix} \text{Hans} \qquad D\vec{H} = \begin{pmatrix} 30 & -10 & 20 \\ -40 & 10 & 5 \\ -10 & 0 & 10 \end{pmatrix} \frac{\text{Hans}}{\text{lightyear}}$$

a. What does he measure as the time rate of change the hyperon field, $\frac{d\vec{H}}{dt}$?

$$\frac{d\vec{H}}{dt} = D\vec{H}\,\vec{v} = \begin{pmatrix} 30 & -10 & 20 \\ -40 & 10 & 5 \\ -10 & 0 & 10 \end{pmatrix} \begin{pmatrix} .1 \\ .3 \\ .2 \end{pmatrix} = \begin{pmatrix} 3 - 3 + 4 \\ -4 + 3 + 1 \\ -1 + 0 + 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$$

b. Using a linear approximation, what would he expect the hyperon field to be at $t = 22$?

$$\vec{H}_{\tan}(t) = \vec{H}(a) + \frac{d\vec{H}}{dt}(t - a)$$
$$\vec{H}_{\tan}(22) = \begin{pmatrix} 200 \\ 150 \\ 300 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}(22 - 20) = \begin{pmatrix} 208 \\ 150 \\ 302 \end{pmatrix}$$

OR $\quad \vec{r}_{\tan}(22) = \vec{r}_{\tan}(20) + \vec{v}(20)(22 - 20) = (20, 10, 30) + 2(.1, .3, .2) = (20.2, 10.6, 30.4)$

$$\vec{H}_{\tan}(\vec{r}_{\tan}(t)) = \vec{H}(\vec{r}(a)) + D\vec{H}(\vec{r}(t) - \vec{r}(a))$$
$$\vec{H}_{\tan}(\vec{r}_{\tan}(22)) = \begin{pmatrix} 200 \\ 150 \\ 300 \end{pmatrix} + \begin{pmatrix} 30 & -10 & 20 \\ -40 & 10 & 5 \\ -10 & 0 & 10 \end{pmatrix} \begin{pmatrix} .2 \\ .6 \\ .4 \end{pmatrix} = \begin{pmatrix} 208 \\ 150 \\ 302 \end{pmatrix}$$

7. (10 points) Consider the vector space $\mathbf{R}^+$ of all positive real numbers with the operations of

Vector Addition: $x \oplus y = xy$ (real number addition)

Scalar Multiplication: $\alpha \odot x = x^\alpha$ (real number exponentiation)

Translate each of the following statements into ordinary arithmetic.

a. For all x we have $0 \odot x = \vec{0}$.

For all x we have $x^0 = 1$.

b. For all a we have $a \odot \vec{0} = \vec{0}$.

For all a we have $1^a = 1$.

c. If $a \odot x = \vec{0}$ then either $a = 0$ or $x = \vec{0}$.

If $x^a = 1$ then either $a = 0$ or $x = 1$.

8. (10 points) Consider the linear function $L : R^3 \to R^2$ given by

$$L(\vec{u}) = \begin{pmatrix} \int_0^1 (u_1 + u_2 x + u_3 x^2)\,dx \\ \frac{d}{dx}(u_1 + u_2 x + u_3 x^2)\Big|_{x=1} \end{pmatrix}$$

Find the matrix A of the linear function, so that you can rewrite it as

$$L(\vec{u}) = A\vec{u}$$

$$L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \int_0^1 (1)\,dx \\ \frac{d}{dx}(1)\Big|_{x=1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \int_0^1 (x)\,dx \\ \frac{d}{dx}(x)\Big|_{x=1} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 2 \end{pmatrix}$$

$$L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \int_0^1 (x^2)\,dx \\ \frac{d}{dx}(x^2)\Big|_{x=1} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 2 \end{pmatrix}$$

OR

$$L(\vec{u}) = \begin{pmatrix} \int_0^1 (u_1 + u_2 x + u_3 x^2)\,dx \\ \frac{d}{dx}(u_1 + u_2 x + u_3 x^2)\Big|_{x=1} \end{pmatrix} = \begin{pmatrix} \left[u_1 x + u_2 \frac{x^2}{2} + u_3 \frac{x^3}{3}\right]_0^1 \\ (u_2 + 2u_3 x)|_{x=1} \end{pmatrix}$$

$$= \begin{pmatrix} u_1 + u_2 \frac{1}{2} + u_3 \frac{1}{3} \\ u_2 + 2u_3 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 2 \end{pmatrix}$$