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1	/60
2	/20
3	/20

MATH 311 Exam 2 Spring 2001
Section 200 Solutions P. Yasskin

1. (60 points) Let P_n be the vector space of polynomials of degree $\le n$. Consider the linear map $I : P_1 \to P_2$ given by

$$I(p)(x) = 2\int_1^x p(t)\,dt$$

Hint: For some parts it may be useful to write $p(t) = a + bt \in P_1$ and/or $q(x) = \alpha + \beta x + \gamma x^2 \in P_2$.

a. (3) Identify the domain of I, a basis for the domain, and the dimension of the domain.

$Dom(I) = P_1 \qquad basis = \{e_1 = 1, \quad e_2 = t\} \qquad \dim Dom(I) = 2$

b. (3) Identify the codomain of I, a basis for the codomain, and the dimension of the codomain.

$Codom(I) = P_2 \qquad basis = \{E_1 = 1, \quad E_2 = x, \quad E_3 = x^2\} \qquad \dim Codom(I) = 3$

c. (5) Identify the kernel of I, a basis for the kernel, and the dimension of the kernel.

$2\int_1^x p(t)\,dt = 0$ differentiate: $2p(x) = 0$

$Ker(I) = \{0\} \qquad basis = \{\text{empty}\} \qquad \dim Ker(I) = 0$

OR: $I(a+bt) = 2\int_1^x a + bt\,dt = [2at + bt^2]_1^x = 2ax + bx^2 - 2a - b = 0 \qquad \Rightarrow \qquad a = b = 0$

d. (5) Identify the range of I, a basis for the range, and the dimension of the range.

$q(x) = 2\int_1^x p(t)\,dt$ differentiate: $\frac{dq}{dx} = 2p(x) \qquad \Rightarrow \qquad p(x) = \frac{1}{2}\frac{dq}{dx}$

Check: $I\left(\frac{1}{2}\frac{dq}{dx}\right) = 2\int_1^x \frac{1}{2}\frac{dq}{dt}\,dt = q(x) - q(1)$ This $= q(x)$ only if $q(1) = 0$

$Ran(I) = \left\{q \in P_2 \text{ such that } q(1) = 0\right\}$ If $q(x) = \alpha + \beta x + \gamma x^2$ then $q(1) = \alpha + \beta + \gamma = 0$.

$Ran(I) = \{q = \alpha + \beta x - (\alpha + \beta)x^2\} = \{\alpha(1 - x^2) + \beta(x - x^2)\} = Span\{1 - x^2, x - x^2\}$

$basis = \{1 - x^2, x - x^2\} \qquad \dim Ran(I) = 2$

OR: $I(a + bt) = 2ax + bx^2 - 2a - b = 2a(x - 1) + b(x^2 - 1)$

$Ran(I) = Span\{x - 1, x^2 - 1\} \qquad basis = \{x - 1, x^2 - 1\} \qquad \dim Ran(I) = 2$

e. (2) Is the function I one-to-one? Why?

$Ker(I) = \{0\} \quad \Rightarrow \quad I$ is $1 - 1$

f. (2) Is the function I onto? Why?

I is not onto because $\dim Codom(I) = 3$ but $\dim Ran(I) = 2$

g. (2) Verify the dimensions in a, b, c and d agree with the Nullity-Rank Theorem.

$\dim Ker(I) + \dim Ran(I) = 0 + 2 = 2 = \dim Dom(I)$

h. (5) Find the matrix of I relative to the standard bases: (Call it $\underset{E\leftarrow e}{A}$.)

$$e_1 = 1, \quad e_2 = t \quad \text{for} \quad P_1 \qquad \text{and} \qquad E_1 = 1, \quad E_2 = x, \quad E_3 = x^2 \quad \text{for} \quad P_2$$

$$\begin{aligned} I(e_1) &= I(1) = 2\int_1^x 1\,dt = 2(x-1) = -2E_1 + 2E_2 \\ I(e_2) &= I(t) = 2\int_1^x t\,dt = x^2 - 1 = -E_1 + E_3 \end{aligned} \quad \Rightarrow \quad \underset{E\leftarrow e}{A} = \begin{pmatrix} -2 & -1 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}$$

i. (6) Another basis for P_1 is $f_1 = 1 + 2t, \quad f_2 = 1 + 3t$. Find the change of basis matrices between the e and f bases. (Call them $\underset{f\leftarrow e}{C}$ and $\underset{e\leftarrow f}{C}$.) Be sure to identify which is which!

$$\begin{aligned} f_1 &= 1 + 2t = e_1 + 2e_2 \\ f_2 &= 1 + 3t = e_1 + 3e_2 \end{aligned} \quad \Rightarrow \quad \underset{e\leftarrow f}{C} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \quad \Rightarrow \quad \underset{f\leftarrow e}{C} = \underset{e\leftarrow f}{C}^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$$

j. (6) Consider the polynomial $q = 3 + 4t$. Find $[q]_e$ and $[q]_f$, the components of q relative to the e and f bases, respectively.

$$[q]_e = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad [q]_f = \underset{f\leftarrow e}{C}\,[q]_e = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

k. (3) A polynomial r has components $[r]_f = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ relative to the f basis. What is r?

$$r = 2f_1 + 1f_2 = 2(1 + 2t) + 1(1 + 3t) = 3 + 7t$$

l. (5) Find the matrix of I relative to the f basis for P_1 and the E basis for P_2. (Call it $\underset{E\leftarrow f}{B}$.)

$$\underset{E\leftarrow f}{B} = \underset{E\leftarrow e}{A}\;\underset{e\leftarrow f}{C} = \begin{pmatrix} -2 & -1 \\ 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} -4 & -5 \\ 2 & 2 \\ 2 & 3 \end{pmatrix}$$

m. (5) Find $\underset{E\leftarrow f}{B}$ by a second method.

$$\begin{aligned} I(f_1) &= I(1 + 2t) = 2\int_1^x (1 + 2t)\,dt = 2(x-1) + 2(x^2-1) = -4E_1 + 2E_2 + 2E_3 \\ I(f_2) &= I(1 + 3t) = 2\int_1^x (1 + 3t)\,dt = 2(x-1) + 3(x^2-1) = -5E_1 + 2E_2 + 3E_3 \end{aligned}$$

$$\Rightarrow \quad \underset{E\leftarrow f}{B} = \begin{pmatrix} -4 & -5 \\ 2 & 2 \\ 2 & 3 \end{pmatrix}$$

n. (6) A polynomial r has components $[r]_f = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ relative to the f basis. Find $[I(r)]_E$, the components of $I(r)$ relative to the E basis. What is $I(r)$?

$$[I(r)]_E = \underset{E\leftarrow f}{B}\,[r]_f = \begin{pmatrix} -4 & -5 \\ 2 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -13 \\ 6 \\ 7 \end{pmatrix} \qquad I(r) = -13 + 6x + 7x^2$$

o. (2) Find $I(r)$ by a second method.

$$I(r) = I(3+7t) = 2\int_1^x (3+7t)\,dt = 6(x-1) + 7(x^2-1) = -13 + 6x + 7x^2$$

2. (20 points) Consider a linear map $L : \mathbf{R}^n \to \mathbf{R}^p$ whose matrix is $A = \begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -4 & 1 & 2 \\ 0 & 0 & 1 & -4 \end{pmatrix}$.

a. (2) What are n and p?

$n = 4 \qquad p = 3$

b. (6) Identify the kernel of L, a basis for the kernel, and the dimension of the kernel.

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 2 & -4 & 1 & 2 & 0 \\ 0 & 0 & 1 & -4 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & -4 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2s - 3t \\ s \\ 4t \\ t \end{pmatrix} \Rightarrow Ker(L) = Span\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\}$$

$$basis = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 4 \\ 1 \end{pmatrix} \right\} \qquad \dim Ker(L) = 2$$

c. (6) Identify the range of L, a basis for the range, and the dimension of the range.

$$Ran(L) = Span\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} \right\}$$

$$= Span\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} \right\}$$ Are they independent?

$$a\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 1 & -4 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -3t \\ 4t \\ t \end{pmatrix}$$

Not independent

$$Ran(L) = Span\left\{\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\} \quad basis = \left\{\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\} \quad \dim Ran(L) = 2$$

d. (2) Is the function L one-to-one? Why?

$Ker(L) \neq \{0\} \quad \Rightarrow \quad L$ is not $1-1$

e. (2) Is the function L onto? Why?

I is not onto because $\dim Codom(I) = 3$ but $\dim Ran(I) = 2$

f. (2) Verify the dimensions in a, b and c agree with the Nullity-Rank Theorem.

$\dim Ker(L) + \dim Ran(L) = 2 + 2 = 4 = \dim Dom(L)$

3. $(20$ points$)$ Consider the parabolic coordinate system

$$x = u^2 - v^2 \qquad y = 2uv$$

a. (4) Describe the u-coordinate curve for which $v = 2$.
(Give an xy-equation and describe the shape.)

If $v = 2$, then $x = u^2 - 4 \quad y = 4u$. So $x = \frac{y^2}{16} - 4$. This is a parabola which opens to the right.

b. (4) Find $\vec{e}_u$, the vector tangent to the u-curve at the point $(u,v) = (3,2)$.

$\vec{e}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\right) = (2u, 2v) \qquad \vec{e}_u\big|_{(3,2)} = (6,4)$

c. (4) Describe the v-coordinate curve for which $u = 3$.
(Give an xy-equation and describe the shape.)

If $u = 3$, then $x = 9 - v^2 \quad y = 6v$. So $x = 9 - \frac{y^2}{36}$. This is a parabola which opens to the left.

d. (4) Find $\vec{e}_v$, the vector tangent to the v-curve at the point $(u,v) = (3,2)$.

$\vec{e}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\right) = (-2v, 2u) \qquad \vec{e}_v\big|_{(3,2)} = (-4,6)$

e. (4) Compute $|\vec{e}_u|$, $|\vec{e}_v|$ and $\vec{e}_u \cdot \vec{e}_v$. Find the angle between $\vec{e}_u$ and $\vec{e}_v$.

$|\vec{e}_u| = \sqrt{36+16} = \sqrt{52}$, $\quad |\vec{e}_v| = \sqrt{16+36} = \sqrt{52}$ and $\vec{e}_u \cdot \vec{e}_v = -24 + 24 = 0$. The angle is $90°$.