Name____________________________ ID______________

1	/20	3	/30
2	/10	4	/40

MATH 311 Exam 3 Spring 2001

Section 200 Solutions P. Yasskin

1. (20 points) Let V be the vector space of functions spanned by $v_1 = t$ and $v_2 = t^2$ with the inner product

$$\langle f,g\rangle = \int_0^1 f(t)\,g(t)\,dt$$

a. (10) Find $\cos\theta$ where θ is the angle between v_1 and v_2.

$$\langle v_1,v_1\rangle = \int_0^1 t\cdot t\,dt = \frac{1}{3} \qquad \langle v_2,v_2\rangle = \int_0^1 t^2\cdot t^2\,dt = \frac{1}{5} = \qquad \langle v_1,v_2\rangle = \int_0^1 t\cdot t^2\,dt = \frac{1}{4}$$

$$\cos\theta = \frac{\langle v_1,v_2\rangle}{\|v_1\|\,\|v_2\|} = \frac{\frac{1}{4}}{\sqrt{\frac{1}{3}}\sqrt{\frac{1}{5}}} = \frac{\sqrt{15}}{4}$$

b. (10) Apply the Gram-Schmidt procedure to $\{v_1,v_2\}$ to produce an orthonormal basis $\{u_1,u_2\}$.

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{t}{\sqrt{\frac{1}{3}}} = \sqrt{3}\,t$$

$$\langle v_2,u_1\rangle = \int_0^1 t^2\sqrt{3}\,t\,dt = \frac{\sqrt{3}}{4} \qquad w_2 = v_2 - \langle v_2,u_1\rangle u_1 = t^2 - \frac{\sqrt{3}}{4}\sqrt{3}\,t = t^2 - \frac{3}{4}t$$

$$\langle w_2,w_2\rangle = \int_0^1 \left(t^2 - \frac{3}{4}t\right)^2 dt = \frac{1}{80}$$

$$u_2 = \frac{w_2}{\|w_2\|} = \sqrt{80}\left(t^2 - \frac{3}{4}t\right) = \sqrt{5}\,(4t^2 - 3t)$$

2. (10 points) In $\mathbf{R}^5$ find the volume of the parallepiped with edges

$$\vec{a} = (1,0,2,0,1)$$
$$\vec{b} = (0,2,1,0,-1)$$
$$\vec{c} = (-1,0,0,2,1)$$

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & -1 \\ -1 & 0 & 0 & 2 & 1 \end{pmatrix} \qquad AA^{\mathsf{T}} = \begin{pmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & -1 \\ -1 & 0 & 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 1 & 6 & -1 \\ 0 & -1 & 6 \end{pmatrix}$$

$$|AA^{\mathsf{T}}| = 6^3 - 6 - 6 = 204 \qquad V = \sqrt{|AA^{\mathsf{T}}|} = \sqrt{204}$$

3. (30 points) A paraboloid in $\mathbf{R}^4$ with coordinates (w,x,y,z), may be parametrized by

$$(w,x,y,z) = \vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2, r^2)$$

for $0 \le r \le \sqrt{3}$ and $0 \le \theta \le 2\pi$.

a. (10) Find the area of the surface.

$$A = \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 2r & 2r \\ -r\sin\theta & r\cos\theta & 0 & 0 \end{pmatrix}$$

$$AA^{\mathsf{T}} = \begin{pmatrix} \vec{e}_r \cdot \vec{e}_r & \vec{e}_r \cdot \vec{e}_\theta \\ \vec{e}_\theta \cdot \vec{e}_r & \vec{e}_\theta \cdot \vec{e}_\theta \end{pmatrix} = \begin{pmatrix} c^2\theta + s^2\theta + 4r^2 + 4r^2 & -rc\theta s\theta + rs\theta c\theta \\ -rc\theta s\theta + rs\theta c\theta & r^2 s^2\theta + r^2 c^2\theta \end{pmatrix} = \begin{pmatrix} 1+8r^2 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$J = \sqrt{\det AA^{\mathsf{T}}} = \sqrt{(1+8r^2)(r^2)} = r\sqrt{1+8r^2}$$

$$Area = \iint 1\,dS = \int_0^{2\pi}\int_0^{\sqrt{3}} r\sqrt{1+8r^2}\,dr\,d\theta = 2\pi\int_0^{\sqrt{3}} r\sqrt{1+8r^2}\,dr = 2\pi\left[\frac{1}{24}(1+8r^2)^{3/2}\right]_0^{\sqrt{3}}$$

$$= \frac{\pi}{12}[125-1] = \frac{31\pi}{3}$$

b. (10) Compute $P = \iint \sqrt{1+8w^2+8x^2}\,dS$ over the surface.

$$\sqrt{1+8w^2+8x^2} = \sqrt{1+8r^2\cos^2\theta + 8r^2\sin^2\theta} = \sqrt{1+8r^2}$$

$$P = \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{1+8r^2}\,r\sqrt{1+8r^2}\,dr\,d\theta = 2\pi\int_0^{\sqrt{3}} r(1+8r^2)\,dr = 2\pi\left[\frac{r^2}{2}+2r^4\right]_0^{\sqrt{3}} = 2\pi\left[\frac{3}{2}+18\right] = 39\pi$$

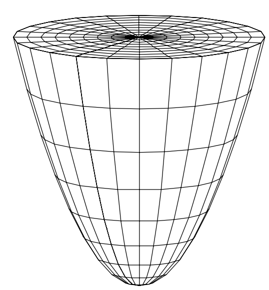
c. (10) Compute $I = \iint (xy\,dw\,dz - wz\,dx\,dy)$ over the surface.

$$w = r\cos\theta, \qquad x = r\sin\theta, \qquad y = r^2, \qquad z = r^2$$

$$\frac{\partial(w,z)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ 2r & 0 \end{vmatrix} = 2r^2\sin\theta \qquad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \sin\theta & r\cos\theta \\ 2r & 0 \end{vmatrix} = -2r^2\cos\theta$$

$$I = \int_0^{2\pi}\int_0^{\sqrt{3}} (r^3\sin\theta(2r^2\sin\theta) - r^3\cos\theta(-2r^2\cos\theta))\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} 2r^5\,dr\,d\theta = 2\pi\left[\frac{r^6}{3}\right]_0^{\sqrt{3}} = 18\pi$$

4. (40 points) The solid paraboloid V at the right is given by $x^2+y^2 \le z \le 4$.
It's boundary (denoted by ∂V) has two parts:
The paraboloid P given by $z = x^2+y^2$ for $z \le 4$.
The disk D given by $x^2+y^2 \le 4$ with $z = 4$.
Let $\vec{G} = (xz^2, yz^2, z(x^2+y^2))$.



a. (5) Compute $\vec{\nabla} \cdot \vec{G}$.

$$\vec{\nabla}\cdot\vec{G} = \frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(z(x^2+y^2)) = 2z^2 + x^2 + y^2$$

b. (10) Compute $\iiint\limits_V \vec{\nabla} \cdot \vec{G}\, dV$ over the solid paraboloid V.

HINT: Use cylindrical coordinates.

In cylindrical coordinates: $\vec{\nabla} \cdot \vec{G} = 2z^2 + r^2 \qquad dV = r\,dr\,d\theta\,dz$

$$\iiint\limits_V \vec{\nabla} \cdot \vec{G}\, dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 (2z^2 + r^2)\, r\, dz\, dr\, d\theta = 2\pi \int_0^2 \left[2\frac{z^3}{3} + r^2 z \right]_{r^2}^4 r\, dr$$

$$= 2\pi \int_0^2 \left(\left[2\frac{64}{3} + r^2 4 \right] - \left[2\frac{r^6}{3} + r^4 \right] \right) r\, dr = 2\pi \left[\frac{128}{3}\frac{r^2}{2} + r^4 - \frac{2}{3}\frac{r^8}{8} - \frac{r^6}{6} \right]_0^2$$

$$= 2\pi \left(\frac{256}{3} + 16 - \frac{64}{3} - \frac{32}{3} \right) = \frac{416}{3}\pi$$

c. (15) Compute $\iint\limits_{P\downarrow} \vec{G} \cdot d\vec{S}$ over the paraboloid P with normal pointing DOWN and OUT.

HINT: Parametrize the paraboloid with coordinates (r, θ).

The paraboloid may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$

$\vec{e}_r = (\cos\theta, \sin\theta, 2r) \qquad \vec{e}_\theta = (-r\sin\theta, r\cos\theta, 0) \qquad \vec{N} = \vec{e}_r \times \vec{e}_\theta = (-2r^2\cos\theta, -2r^2\sin\theta, r)$

This normal points UP and IN. So reverse it: $\vec{N} = (2r^2\cos\theta, 2r^2\sin\theta, -r)$

$\vec{G} = (xz^2, yz^2, z(x^2+y^2)) = (r^5\cos\theta, r^5\sin\theta, r^4) \qquad \vec{G} \cdot \vec{N} = 2r^7 - r^5$

$$\iint\limits_{P\downarrow} \vec{G} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 (2r^7 - r^5)\, dr\, d\theta = 2\pi \left[\frac{r^8}{4} - \frac{r^6}{6} \right]_0^2 = 2\pi \left(64 - \frac{32}{3} \right) = \frac{320}{3}\pi$$

d. (5) Compute $\iint\limits_{D\uparrow} \vec{G} \cdot d\vec{S}$ over the disk D with normal pointing UP.

HINT: Parametrize the disk with coordinates (r, θ).

The disk may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 4)$

$\vec{e}_r = (\cos\theta, \sin\theta, 0) \qquad \vec{e}_\theta = (-r\sin\theta, r\cos\theta, 0) \qquad \vec{N} = \vec{e}_r \times \vec{e}_\theta = (0, 0, r)$ which points UP.

$\vec{G} = (xz^2, yz^2, z(x^2+y^2)) = (16r\cos\theta, 16r\sin\theta, 4r^2) \qquad \vec{G} \cdot \vec{N} = 4r^3$

$$\iint\limits_{D\uparrow} \vec{G} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 (4r^3)\, dr\, d\theta = 2\pi [r^4]_0^2 = 32\pi$$

e. (5) Compute $\iint\limits_{\partial V} \vec{G} \cdot d\vec{S} = \iint\limits_{P\downarrow} \vec{G} \cdot d\vec{S} + \iint\limits_{D\uparrow} \vec{G} \cdot d\vec{S}$

(Note: By Gauss' Theorem, the answers to (b) and (e) should be equal.)

$$\iint\limits_{\partial V} \vec{G} \cdot d\vec{S} = \frac{320}{3}\pi + 32\pi = \frac{416}{3}\pi$$