

Name \_\_\_\_\_ ID \_\_\_\_\_

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MATH 311                      Exam 3                      Fall 2002  
 Section 200                    Solutions                      P. Yasskin

Multiple Choice: (10 points each)    Work Out: (15 points each)    Extra Credit: (10 points)

1. If  $\vec{\nabla} \times \vec{F} = (2x^2y - x^2, y^2 - 2xy^2, 2xz - 2yz)$ , then  $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) =$

- a.  $(4xy - 2x, 4xy - 2y, 2x - 2y)$
- b.  $(4xy - 2x, 2y - 4xy, 2x - 2y)$
- c.  $8xy - 4y$
- d. 0      correctchoice
- e.  $2x^2 - 2y^2$

For any vector field  $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$

2. If  $\vec{\nabla} \times \vec{F} = (2x^2y - x^2, y^2 - 2xy^2, 2xz - 2yz)$ , then  $\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) =$

- a.  $(4xy - 2x, 4xy - 2y, 2x - 2y)$
- b.  $(4xy - 2x, 2y - 4xy, 2x - 2y)$
- c.  $(-2z, -2z, -2x^2 - 2y^2)$       correctchoice
- d.  $(-2z, 2z, -2x^2 - 2y^2)$
- e.  $-2x^2 - 2y^2$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2x^2y - x^2 & y^2 - 2xy^2 & 2xz - 2yz \end{vmatrix} = \hat{i}(-2z) - \hat{j}(2z) + \hat{k}(-2y^2 - 2x^2)$$

3. If  $\vec{G} = (2x^2y - x^2, y^2 - 2xy^2, 2xz - 2yz)$ , then  $\vec{G} = \vec{\nabla}g$  where  $g(0, 1, 1) - g(0, 1, 0) =$

- a. -2
- b. -1
- c. 0
- d. 1
- e. The scalar potential  $g$  does not exist.      correctchoice

$$\text{Since } \vec{\nabla} \times \vec{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2x^2y - x^2 & y^2 - 2xy^2 & 2xz - 2yz \end{vmatrix} = \hat{i}(-2z) - \hat{j}(2z) + \hat{k}(-2y^2 - 2x^2) \neq (0, 0, 0),$$

the scalar potential  $g$  does not exist.

4. Compute  $\oint (y+z) dx + (x+z) dy + (x+y) dz$  clockwise around the circle  $x^2 + y^2 = 9$  with  $z = 5$ .

HINT: Use a theorem.

- a.  $-18\pi$
- b.  $-9\pi$
- c. 0 correct choice
- d.  $9\pi$
- e.  $18\pi$

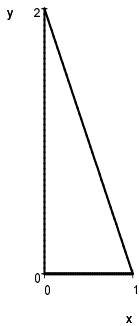
Let  $\vec{F} = (y+z, x+z, x+y)$ . Then  $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y+z & x+z & x+y \end{vmatrix} = (0,0,0)$ .

By Stokes' Theorem,  $\oint \vec{F} \cdot d\vec{S} = \iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = 0$

where the surface integral is over the disk bounded by the circle.

5. Compute  $\oint_{\partial T} (xy) dx + (xy) dy$  counterclockwise around the boundary of the triangle with vertices  $(0,0)$ ,  $(1,0)$  and  $(0,2)$ .

By Green's theorem,



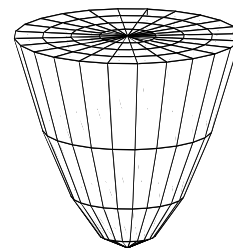
$$\oint_{\partial T} (xy) dx + (xy) dy = \iint_T \left( \frac{\partial(xy)}{\partial x} - \frac{\partial(xy)}{\partial y} \right) dx dy$$

$$= \int_0^1 \int_0^{2-2x} (y-x) dy dx = \int_0^1 \left[ \frac{y^2}{2} - xy \right]_0^{2-2x} dx$$

$$= \int_0^1 \left[ \frac{(2-2x)^2}{2} - x(2-2x) \right] dx = \int_0^1 (2 - 4x + 2x^2 - 2x + 2x^2) dx$$

$$= \int_0^1 (2 - 6x + 4x^2) dx = \left[ 2x - 3x^2 + 4 \frac{x^3}{3} \right]_0^1 = \left[ 2 - 3 + \frac{4}{3} \right] = \frac{1}{3}$$

6. Compute  $\iint_{\partial P} \vec{E} \cdot d\vec{S}$  for  $\vec{E} = (xz, yz, z^2)$  over the **complete** surface of the solid paraboloid  $P$  given by  $x^2 + y^2 \leq z \leq 4$  with outward normal.

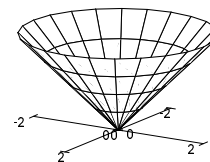


By Gauss' Theorem,  $\iint_{\partial P} \vec{E} \cdot d\vec{S} = \iiint_P \vec{\nabla} \cdot \vec{E} dV = \iiint_P (z + z + 2z) dV = \iiint_P (4z) dV$

In cylindrical coordinates, the paraboloid is  $r^2 \leq z \leq 4$ . So

$$\begin{aligned} \iint_{\partial P} \vec{E} \cdot d\vec{S} &= \int_0^2 \int_0^{2\pi} \int_{r^2}^4 (4z) r dz d\theta dr = 2\pi \int_0^2 [2z^2]_{z=r^2}^4 r dr = 2\pi \int_0^2 (32 - 2r^4) r dr \\ &= 2\pi \left[ 32 \frac{r^2}{2} - 2 \frac{r^6}{6} \right]_0^2 = 2\pi \left( 64 - \frac{64}{3} \right) = 128\pi \left( 1 - \frac{1}{3} \right) = \frac{256\pi}{3} \end{aligned}$$

7. The cone  $z = \sqrt{x^2 + y^2}$  for  $z \leq 2$  is shown at the right. Find the mass and center of mass of the cone if its surface density is given by  $\delta = x^2 + y^2$ .



$$\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, r) \quad \vec{e}_r = (\cos \theta, \sin \theta, 1) \quad \vec{e}_\theta = (-r \sin \theta, r \cos \theta, 0)$$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = (-r \cos \theta, -r \sin \theta, r) \quad |\vec{N}| = \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta + r^2} = \sqrt{2} r$$

$$\delta = x^2 + y^2 = r^2$$

$$M = \iint_C \delta dS = \iint_C \delta(\vec{R}(r, \theta)) |\vec{N}| dr d\theta = \int_0^{2\pi} \int_0^2 r^2 \sqrt{2} r dr d\theta = 2\pi \sqrt{2} \frac{r^4}{4} \Big|_0^2 = 8\pi \sqrt{2}$$

By symmetry,  $\bar{x} = \bar{y} = 0$ .

$$z\text{-mom} = \iint_C z \delta dS = \iint_C z \delta(\vec{R}(r, \theta)) |\vec{N}| dr d\theta = \int_0^{2\pi} \int_0^2 r^3 \sqrt{2} r dr d\theta = 2\pi \sqrt{2} \frac{r^5}{5} \Big|_0^2 = \frac{64\pi \sqrt{2}}{5}$$

$$\bar{z} = \frac{z\text{-mom}}{M} = \frac{64\pi \sqrt{2}}{5 \cdot 8\pi \sqrt{2}} = \frac{8}{5}$$

8. Compute  $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$  for  $\vec{F} = (x^2y, -x^3, z^2)$

over the piece of the sphere  $x^2 + y^2 + z^2 = 25$

for  $0 \leq z \leq 4$  with normal pointing away from the  $z$ -axis.



Hint: Parametrize the upper and lower edges.

By Stokes' Theorem 
$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s} = \oint_{\text{upper}} \vec{F} \cdot d\vec{s} + \oint_{\text{lower}} \vec{F} \cdot d\vec{s}$$

By the right hand rule, since the normal points outward, the **upper** circle must be traversed **clockwise** while the **lower** circle must be traversed **counterclockwise** as seen from the positive  $z$ -axis. We compute each line integral:

Upper Circle:  $z = 4 \quad x^2 + y^2 = 25 - z^2 = 25 - 16 = 9$

$\vec{r}(t) = (3 \cos t, 3 \sin t, 4) \quad \vec{v} = (-3 \sin t, 3 \cos t, 0)$

This is clockwise, so we reverse the velocity:  $\vec{v} = (3 \sin t, -3 \cos t, 0)$

$\vec{F} = (x^2y, -x^3, z^2) = (27 \cos^2 t \sin t, -27 \cos^3 t, 16)$

$$\oint_{\text{upper}} \vec{F} \cdot d\vec{s} = \oint_{\text{upper}} \vec{F} \cdot \vec{v} dt = \int_0^{2\pi} (81 \cos^2 t \sin^2 t + 81 \cos^4 t) dt = 81 \int_0^{2\pi} \cos^2 t (\sin^2 t + \cos^2 t) dt$$

$$= 81 \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt = \frac{81}{2} \left[ t + \frac{\sin 2t}{2} \right]_0^{2\pi} = 81\pi$$

Lower Circle:  $z = 0 \quad x^2 + y^2 = 25 - z^2 = 25$

$\vec{r}(t) = (5 \cos t, 5 \sin t, 0) \quad \vec{v} = (-5 \sin t, 5 \cos t, 0)$

This is counterclockwise, so we do not need to reverse the velocity.

$\vec{F} = (x^2y, -x^3, z^2) = (125 \cos^2 t \sin t, -125 \cos^3 t, 0)$

$$\oint_{\text{lower}} \vec{F} \cdot d\vec{s} = \oint_{\text{lower}} \vec{F} \cdot \vec{v} dt = \int_0^{2\pi} (-625 \cos^2 t \sin^2 t - 625 \cos^4 t) dt = -625 \int_0^{2\pi} \cos^2 t (\sin^2 t + \cos^2 t) dt$$

$$= -625 \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt = \frac{-625}{2} \left[ t + \frac{\sin 2t}{2} \right]_0^{2\pi} = -625\pi$$

Total Boundary:

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = 81\pi - 625\pi = -544\pi$$

**Extra Credit** Redo #8 but compute  $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$  directly as a surface integral.

Use spherical coordinates:

$$\vec{R}(\varphi, \theta) = (5 \sin \varphi \cos \theta, 5 \sin \varphi \sin \theta, 5 \cos \varphi) \quad \arccos \frac{4}{5} \leq \varphi \leq \frac{\pi}{2} \quad 0 \leq \theta \leq 2\pi$$

$$\vec{e}_\varphi = (5 \cos \varphi \cos \theta, 5 \cos \varphi \sin \theta, -5 \sin \varphi) \quad \vec{e}_\theta = (-5 \sin \varphi \sin \theta, 5 \sin \varphi \cos \theta, 0)$$

$$\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = (25 \sin^2 \varphi \cos \theta, 25 \sin^2 \varphi \sin \theta, 25 \sin \varphi \cos \varphi)$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 y & -x^3 & z^2 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(-3x^2 - x^2) = (0, 0, -4x^2) = (0, 0, -100 \sin^2 \varphi \cos^2 \theta)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = -2500 \sin^3 \varphi \cos \varphi \cos^2 \theta$$

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_S \vec{\nabla} \times \vec{F} \cdot \vec{N} d\varphi d\theta = \int_0^{2\pi} \int_{\arccos 4/5}^{\pi/2} -2500 \sin^3 \varphi \cos \varphi \cos^2 \theta d\varphi d\theta$$

$$= -2500 \int_0^{2\pi} \cos^2 \theta d\theta \int_{\arccos 4/5}^{\pi/2} \sin^3 \varphi \cos \varphi d\varphi = -2500 \left[ \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{2\pi} \left[ \frac{\sin^4 \varphi}{4} \right]_{\arccos 4/5}^{\pi/2}$$

$$= -\frac{2500}{4} [\pi] [1 - \sin^4 \arccos 4/5] = -\frac{2500\pi}{4} \left[ 1 - \left( \frac{3}{5} \right)^4 \right] = -\frac{2500\pi}{4} \left( \frac{625 - 81}{625} \right) = -544\pi$$