Name________________________ ID__________

1-3	/30	6	/20
4	/20	7	/10
5	/20	8	/10

MATH 311 Final Exam Fall 2002
Section 200 Solutions P. Yasskin

Multiple Choice: (10 points each) Work Out: (points indicated) Extra Credit: (10 points)

1. (10 points) If $L : P_2 \to \mathbb{R}$ is a linear function satisfying

$$L(1+x+x^2) = 2 \qquad L(x+x^2) = -1 \qquad \text{and} \qquad L(x^2) = 3$$

find $L(2+3x+5x^2)$.

a. 16
b. 9 correctchoice
c. 4
d. 0
e. -4

$2+3x+5x^2 = a(1+x+x^2)+b(x+x^2)+c(x^2)$
By inspection or row operations, $a=2$, $b=1$, $c=2$. So by linearity
$L(2+3x+5x^2) = L(2(1+x+x^2)+(x+x^2)+2(x^2))$
$= 2L(1+x+x^2)+L(x+x^2)+2L(x^2) = 2(2)+(-1)+2(3) = 9$

2. (10 points) Find the plane tangent to the hyperbolic paraboloid $x-yz=0$ at the point $P=(6,3,2)$. Which of the following points does **not** lie on this plane?

a. $(-6,0,0)$
b. $(0,3,0)$
c. $(0,0,2)$
d. $(-1,1,1)$
e. $(1,-1,-1)$ correctchoice

The hyperbolic paraboloid is a level surface of the function $g=x-yz$.
Its gradient is $\vec{\nabla} g = (1,-z,-y)$.
So the normal to the surface at P is $\vec{N} = \vec{\nabla} g\big|_{(6,3,2)} = (1,-2,-3)$.
So the tangent plane is $\vec{N}\cdot X = \vec{N}\cdot P$, or $x-2y-3z = 6-2\cdot 3-3\cdot 2 = -6$.
Plugging in each point, we find $(1,-1,-1)$ is not a solution.

3. (10 points) Duke Skywater is flying the Millenium Eagle through a polaron field. His galactic coordinates are $(2300, 4200, 1600)$ measured in lightseconds and his velocity is $\vec{v} = (.2, .3, .4)$ measured in lightseconds per second. He measures the strength of the polaron field is $p = 274$ milliwookies and its gradient is $\vec{\nabla} p = (3, 2, 2)$ milliwookies per lightsecond. Assuming a linear approximation for the polaron field and that his velocity is constant, how many seconds will Duke need to wait until the polaron field has grown to 286 milliwookies?

a. 2
b. 3
c. 4
d. 6 correctchoice
e. 12

The derivative along Duke's path is

$$\frac{dp}{dt} = \vec{v} \cdot \vec{\nabla} p = (.2, .3, .4) \frac{\text{lightseconds}}{\text{second}} \cdot (3, 2, 2) \frac{\text{milliwookies}}{\text{lightsecond}} = .6 + .6 + .8 = 2 \frac{\text{milliwookies}}{\text{second}}$$

So the polaron field increases 2 milliwookies each second.
To increase 12 milliwookies, it will take 6 seconds.

4. (20 points) Consider the linear map $f : \mathbb{R}^5 \to \mathbb{R}^3$ given by $f(\vec{x}) = A\vec{x}$ where

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 & 1 \\ 1 & 2 & 0 & 3 & -1 \\ -2 & -4 & 0 & -6 & 2 \end{pmatrix}.$$

When necessary, let $\vec{x} \in \mathbb{R}^5$ be $\vec{x} = \begin{pmatrix} r \\ s \\ t \\ u \\ v \end{pmatrix}$ and $\vec{z} \in \mathbb{R}^3$ be $\vec{z} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

a. (2) Identify the domain and give its dimension.

$Dom(f) = \mathbb{R}^5 \qquad \dim Dom(f) = 5$

b. (2) Identify the codomain and give its dimension.

$Codom(f) = \mathbb{R}^3 \qquad \dim Codom(f) = 3$

c. (2) Verify that f is linear.

$f(\vec{x} + \vec{y}) = A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = f(\vec{x}) + f(\vec{y})$ additive
$f(a\vec{x}) = A(a\vec{x}) = a(A\vec{x}) = af(\vec{x})$ scalar multiplicative

d. (4) Find the kernel of f. Write it as a $Span$ and give a basis and its dimension.

$Ker(f) = \{\vec{x} \mid f(\vec{x}) = \vec{0}\}$ Solve

$$\begin{pmatrix} 1 & 2 & -1 & 3 & 1 \\ 1 & 2 & 0 & 3 & -1 \\ -2 & -4 & 0 & -6 & 2 \end{pmatrix} \begin{pmatrix} r \\ s \\ t \\ u \\ v \end{pmatrix} = \begin{pmatrix} r + 2s - t + 3u + v \\ r + 2s + 3u - v \\ -2r - 4s - 6u + 2v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & -1 & 3 & 1 & 0 \\ 1 & 2 & 0 & 3 & -1 & 0 \\ -2 & -4 & 0 & -6 & 2 & 0 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ R_3 + 2R_1 \end{array} \Rightarrow \left(\begin{array}{ccccc|c} 1 & 2 & -1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & -2 & 0 & 4 & 0 \end{array}\right) \begin{array}{l} R_1 + R_2 \\ \\ R_3 + 2R_2 \end{array}$$

$$\Rightarrow \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} r \\ s \\ t \\ u \\ v \end{pmatrix} = \begin{pmatrix} -2s - 3u + v \\ s \\ 2v \\ u \\ v \end{pmatrix}$$

$$Ker(f) = \left\{ \begin{pmatrix} -2s - 3u + v \\ s \\ 2v \\ u \\ v \end{pmatrix} \right\} = \left\{ s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + u\begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + v\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$= Span\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Basis is: $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

$\dim Ker(f) = 3$

e. (4) Find the image of f. Write it as a $Span$ and give a basis and its dimension.

$$\mathrm{Im}(f) = \{f(\vec{x})\} = \left\{ \begin{pmatrix} r + 2s - t + 3u + v \\ r + 2s + 3u - v \\ -2r - 4s - 6u + 2v \end{pmatrix} \right\}$$

$$= \left\{ r\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + s\begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + u\begin{pmatrix} 3 \\ 3 \\ -6 \end{pmatrix} + v\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}$$

$$= Span\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -6 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\} = Span\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

Basis is: $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ $\quad \dim \mathrm{Im}(f) = 2$

f. (2) Verify your answers are consistent with the Nullity-Rank Theorem.

$\dim Ker(f) + \dim \mathrm{Im}(f) = 3 + 2 = 5 = \dim Dom(f)$

g. (2) Is f one-to-one? Why?

f is NOT one-to-one because $Ker(f) \neq \{\vec{0}\}$

h. (2) Is f onto? Why?

f is NOT onto because $\mathrm{Im}(f) \neq Codom(f) = \mathbb{R}^3$

5. $\left(20 \text{ points}\right)$ Consider the vector space P_2 of polynomials of degree ≤ 2.
Consider the bases

$$e_1 = 1 \qquad e_2 = x \qquad e_3 = x^2$$

$$f_1 = 1 + x \qquad f_2 = x \qquad f_3 = -x + x^2$$

Consider the function $L : P_2 \to P_2$ given by

$$L(p) = 2p(0) + p(1)x$$

a. (4) Find the matrix of L relative to the e-basis (on both the domain and the codomain). Call it $\underset{e\leftarrow e}{A}$.

$$\begin{aligned} L(e_1) &= L(1) &&= 2 + x &&= 2e_1 + e_2 \\ L(e_2) &= L(x) &&= x &&= e_2 \\ L(e_3) &= L(x^2) &&= x &&= e_2 \end{aligned} \qquad \underset{e\leftarrow e}{A} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

b. (8) Find the change of basis matrices between the e and f bases. (Call them $\underset{f\leftarrow e}{C}$ and $\underset{e\leftarrow f}{C}$.)

Be sure to identify which is which!

$$\begin{aligned} f_1 &= 1 + x &&= e_1 + e_2 \\ f_2 &= x &&= e_2 \\ f_3 &= -x + x^2 &&= -e_2 + e_3 \end{aligned} \qquad \underset{e\leftarrow f}{C} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad R2 - R1 + R3 \quad \Rightarrow \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \qquad \underset{f\leftarrow e}{C} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

c. (4) Find the matrix of L relative to the f-basis. Call it $\underset{f\leftarrow f}{B}$.

$$\underset{f\leftarrow f}{B} = \underset{f\leftarrow e}{C}\ \underset{e\leftarrow e}{A}\ \underset{e\leftarrow f}{C} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

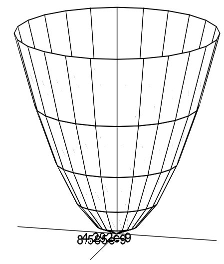
d. (4) Find $\underset{f\leftarrow f}{B}$ by a second method.

Recall $L(p) = 2p(0) + p(1)x$

$$\begin{aligned} L(f_1) &= L(1 + x) &&= 2 + 2x &&= 2f_1 \\ L(f_2) &= L(x) &&= x &&= f_2 \\ L(f_3) &= L(-x + x^2) &&= 0 &&= 0 \end{aligned} \qquad \underset{f\leftarrow f}{B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

6. $\left(20 \text{ points}\right)$ **Stokes' Theorem** states that if S is a nice surface in $\mathbf{R}^3$ and ∂S is its boundary curve traversed counterclockwise as seen from the tip of the normal to S and $\vec{F}$ is a nice vector field on S then

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$$



Verify Stokes' Theorem if

$F = (y, -x, x^2 + y^2)$

and S is the paraboloid $z = x^2 + y^2$ for $z \le 4$

with **normal pointing up and in**.

Remember to check the orientations.

The paraboloid may be parametrized by:

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$

a. (10) Compute $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ using the following steps:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & -x & x^2 + y^2 \end{vmatrix} = i(2y - 0) - j(2x - 0) + k(-1 - 1) = (2y, -2x, -2)$$

$\left(\vec{\nabla} \times \vec{F}\right)\left(\vec{R}(r,\theta)\right) = (2r\sin\theta, -2r\cos\theta, -2)$

$\vec{R}_r = (\cos\theta, \sin\theta, 2r)$

$\vec{R}_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\vec{N} = i(-2r^2\cos\theta) - j(2r^2\sin\theta) + k(r\cos^2\theta + r\sin^2\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$

This is oriented correctly as up and in.

$$\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int\int (-4r^3\sin\theta\cos\theta + 4r^3\sin\theta\cos\theta - 2r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (-2r)\, dr\, d\theta = 2\pi\left[-r^2\right]_0^2 = -8\pi$$

b. (10) Recall $F = (y, -x, x^2 + y^2)$ and S is the paraboloid $z = x^2 + y^2$ for $z \le 4$ with **normal pointing up and in**. Compute $\oint_{\partial S} \vec{F} \cdot d\vec{s}$ using the following steps:

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 4)$

$\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta, 0)$ which is correctly counterclockwise.

$\vec{F}(\vec{r}(\theta)) = (2\sin\theta, -2\cos\theta, 4)$

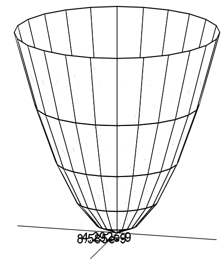
$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} (-4\sin^2\theta - 4\cos^2\theta)\, d\theta = \int_0^{2\pi} (-4)\, d\theta = -8\pi$$

7. (10 points) The paraboloid at the right is the graph of the equation $z = x^2 + y^2$.

It may be parametrized as

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2).$$

Find the area of the paraboloid for $z \le 4$.



$$\vec{R}_r = (\cos\theta, \sin\theta, 2r)$$

$$\vec{R}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$\vec{N} = i(-2r^2\cos\theta) - j(2r^2\sin\theta) + k(r\cos^2\theta + r\sin^2\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$$\left|\vec{N}\right| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$$

$$A = \iint \left|\vec{N}\right| \, dr\, d\theta = \int_0^{2\pi}\int_0^2 r\sqrt{4r^2+1}\, dr\, d\theta = 2\pi\left[\frac{2(4r^2+1)^{3/2}}{3\cdot 8}\right]_0^2 = \frac{\pi}{6}(17^{3/2} - 1)$$

8. (10 points) A paraboloid in $\mathbf{R}^4$ with coordinates (w,x,y,z), may be parametrized by

$$(w,x,y,z) = \vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2, r^2) \quad \text{for } 0 \le r \le 3 \text{ and } 0 \le \theta \le 2\pi.$$

Compute $I = \iint (xz\, dw\, dy - wy\, dx\, dz)$ over the surface.

$$w = r\cos\theta, \qquad x = r\sin\theta, \qquad y = r^2, \qquad z = r^2$$

$$\frac{\partial(w,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ 2r & 0 \end{vmatrix} = 2r^2\sin\theta \qquad \frac{\partial(x,z)}{\partial(r,\theta)} = \begin{vmatrix} \sin\theta & r\cos\theta \\ 2r & 0 \end{vmatrix} = -2r^2\cos\theta$$

$$I = \int_0^{2\pi}\int_0^3 (r^3\sin\theta(2r^2\sin\theta) - r^3\cos\theta(-2r^2\cos\theta))\, dr\, d\theta = \int_0^{2\pi}\int_0^3 2r^5\, dr\, d\theta = 2\pi\left[\frac{r^6}{3}\right]_0^3 = 486\pi$$