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MATH 311 Exam 2 Spring 2003
Section 200 P. Yasskin

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3	/20	6	/15

Throughout the exam, let $(P_2)^2$ be the vector space of ordered pairs of polynomials of degree less than 2. For example,

$$\vec{q} = \begin{pmatrix} 2x-3 \\ 3x+1 \end{pmatrix} \in (P_2)^2 \qquad \text{and} \qquad \vec{q}(2) = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

The standard basis of $(P_2)^2$ is

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

1. $(20$ points$)$ Another basis for $(P_2)^2$ is

$$E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1+x \\ 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 \\ 1+x \end{pmatrix}$$

a. (5) Find the change of basis matrices $\underset{E\leftarrow e}{C}$ and $\underset{e\leftarrow E}{C}$.

b. (5) Find $(\vec{q})_e$ the components of $\vec{q} = \begin{pmatrix} 2x-3 \\ 3x+1 \end{pmatrix}$ relative to the e-basis.

c. (5) Find $(\vec{q})_E$ the components of $\vec{q}$ relative to the E-basis by using the change of basis matrix.

d. (5) If $(\vec{r})_E = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, what is $\vec{r}$?

2. (15 points) Consider the subspace S of $(P_3)^2$ spanned by $\begin{pmatrix} 1+x \\ 1-x \end{pmatrix}$, $\begin{pmatrix} 2+x \\ 2-x \end{pmatrix}$, $\begin{pmatrix} 3+x \\ 3-x \end{pmatrix}$, $\begin{pmatrix} 1-x \\ 1+x \end{pmatrix}$. Pare the spanning set down to a basis for S and find the dimension of S.

3. (20 points) Now consider the linear map $L : (P_2)^2 \to P_2$ given by $L(\vec{p}) = p_1 + p_2$. (Just add the two component polynomials.) For example, if $\vec{q} = \begin{pmatrix} -3+2x \\ 1+3x \end{pmatrix}$ then

$$L(\vec{q}) = L\begin{pmatrix} -3+2x \\ 1+3x \end{pmatrix} = (-3+2x) + (1+3x) = -2+5x$$

a. (5) Find the matrix of L relative to the e-basis on $(P_2)^2$ and the f-basis on P_2 where $f_1 = 1$ and $f_2 = x$. Call it $\underset{f \leftarrow e}{A}$.

b. (5) Find the matrix of L relative to the E-basis on $(P_2)^2$ and the f-basis on P_2 by using the change of basis matrix. Call it $\underset{f \leftarrow E}{B}$.

c. (5) Find the matrix of L relative to the E-basis on $(P_2)^2$ and the f-basis on P_2 from the definition.

d. (5) If $(\vec{r})_E = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, what are $[L(\vec{r})]_f$ and $L(\vec{r})$?

4. (15 points) Again consider the linear map $L : (P_2)^2 \to P_2$ given by $L(\vec{p}) = p_1 + p_2$. When necessary, let $\vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} a + bx \\ c + dx \end{pmatrix}$.

a. (5) Find the kernel of L. Give a basis and the dimension.

b. (5) Find the image of L. Give a basis and the dimension.

c. (2) Is L one-to-one? Why?

d. (2) Is L onto? Why?

e. (1) Check that the Nullity-Rank Theorem is satisfied.

5. $(15$ points$)$ Verify that the following function is an inner product on $(P_2)^2$:

$$\langle\ ,\ \rangle : (P_2)^2 \times (P_2)^2 \to \mathbb{R} \quad \text{given by} \quad \langle \vec{p}, \vec{q} \rangle = \int_{-1}^{1} p_1(x)q_1(x) + p_2(x)q_2(x)\, dx$$

For example, $\left\langle \begin{pmatrix} 1+x \\ 2x \end{pmatrix}, \begin{pmatrix} -x \\ 2-x \end{pmatrix} \right\rangle = \int_{-1}^{1} (1+x)(-x) + (2x)(2-x)\, dx = \int_{-1}^{1} (3x - 3x^2)\, dx = -2$

a. Symmetric:

b. Bilinear:

c. Positive Definite:

6. $\left(15\text{ points}\right)$ Using the inner product of problem 5, find the angle between the vectors $\begin{pmatrix} 1 \\ x \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -x \end{pmatrix}$.