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MATH 311 Exam 2 Spring 2003
Section 200 Solutions P. Yasskin

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Throughout the exam, let $(P_2)^2$ be the vector space of ordered pairs of polynomials of degree less than 2. For example,

$$\vec{q} = \begin{pmatrix} 2x-3 \\ 3x+1 \end{pmatrix} \in (P_2)^2 \qquad \text{and} \qquad \vec{q}(2) = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

The standard basis of $(P_2)^2$ is

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} x \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 \\ x \end{pmatrix}$$

1. $(20$ points$)$ Another basis for $(P_2)^2$ is

$$E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 1+x \\ 0 \end{pmatrix} \quad E_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad E_4 = \begin{pmatrix} 0 \\ 1+x \end{pmatrix}$$

a. (5) Find the change of basis matrices $\underset{E\leftarrow e}{C}$ and $\underset{e\leftarrow E}{C}$.

$$\begin{aligned} E_1 &= e_1 \\ E_2 &= e_1 + e_2 \\ E_3 &= e_3 \\ E_4 &= e_3 + e_4 \end{aligned} \quad \Rightarrow \quad \underset{e\leftarrow E}{C} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} e_1 &= E_1 \\ e_2 &= E_2 - E_1 \\ e_3 &= E_3 \\ e_4 &= E_4 - E_3 \end{aligned} \quad \Rightarrow \quad \underset{E\leftarrow e}{C} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

b. (5) Find $(\vec{q})_e$ the components of $\vec{q} = \begin{pmatrix} 2x-3 \\ 3x+1 \end{pmatrix}$ relative to the e-basis.

$$\vec{q} = -3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} x \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ x \end{pmatrix} \qquad \Rightarrow \qquad (\vec{q})_e = \begin{pmatrix} -3 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

c. (5) Find $(\vec{q})_E$ the components of $\vec{q}$ relative to the E-basis by using the change of basis matrix.

$$(\vec{q})_E = \underset{E\leftarrow e}{C} (\vec{q})_e = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ -2 \\ 3 \end{pmatrix}$$

Check by hooking the components back onto the basis vectors.

$$\vec{q} = -5E_1 + 2E_2 - 2E_3 + 3E_4 = -5\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1+x \\ 0 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1+x \end{pmatrix} = \begin{pmatrix} -3+2x \\ 1+3x \end{pmatrix}$$

d. (5) If $(\vec{r})_E = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, what is $\vec{r}$?

$$\vec{r} = 1E_1 + 2E_2 + 3E_3 + 4E_4 = 1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1+x \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 1+x \end{pmatrix} = \begin{pmatrix} 3+2x \\ 7+4x \end{pmatrix}$$

2. (15 points) Consider the subspace S of $(P_3)^2$ spanned by $\begin{pmatrix} 1+x \\ 1-x \end{pmatrix}$, $\begin{pmatrix} 2+x \\ 2-x \end{pmatrix}$, $\begin{pmatrix} 3+x \\ 3-x \end{pmatrix}$, $\begin{pmatrix} 1-x \\ 1+x \end{pmatrix}$. Pare the spanning set down to a basis for S and find the dimension of S.

$$a\begin{pmatrix} 1+x \\ 1-x \end{pmatrix} + b\begin{pmatrix} 2+x \\ 2-x \end{pmatrix} + c\begin{pmatrix} 3+x \\ 3-x \end{pmatrix} + d\begin{pmatrix} 1-x \\ 1+x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a(1+x)+b(2+x)+c(3+x)+d(1-x) \\ a(1-x)+b(2-x)+c(3-x)+d(1+x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} (a+2b+3c+d)1+(a+b+c-d)x \\ (a+2b+3c+d)1+(-a-b-c+d)x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a+2b+3c+d=0 \\ a+b+c-d=0 \\ a+2b+3c+d=0 \\ -a-b-c+d=0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 1 & 2 & 3 & 1 & 0 \\ -1 & -1 & -1 & 1 & 0 \end{pmatrix} \begin{matrix} \\ R_2-R_1 \\ R_3-R_1 \\ R_4+R_2 \end{matrix} \Rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 & 0 \\ 0 & -1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1+2R_2 \\ -R_2 \\ \\ \\ \end{matrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

a & b are leading variables
c & d are free variables

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} s+3t \\ -2s-2t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

Let $s=1,\ t=0$: $\Rightarrow$ $\begin{pmatrix} 1+x \\ 1-x \end{pmatrix} - 2\begin{pmatrix} 2+x \\ 2-x \end{pmatrix} + \begin{pmatrix} 3+x \\ 3-x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Let $s=0,\ t=1$: $3\begin{pmatrix} 1+x \\ 1-x \end{pmatrix} - 2\begin{pmatrix} 2+x \\ 2-x \end{pmatrix} + \begin{pmatrix} 1-x \\ 1+x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So $\begin{pmatrix} 3+x \\ 3-x \end{pmatrix}$ and $\begin{pmatrix} 1-x \\ 1+x \end{pmatrix}$ can be expressed in terms of $\begin{pmatrix} 1+x \\ 1-x \end{pmatrix}$ and $\begin{pmatrix} 2+x \\ 2-x \end{pmatrix}$.

A basis is $\begin{pmatrix} 1+x \\ 1-x \end{pmatrix}$, $\begin{pmatrix} 2+x \\ 2-x \end{pmatrix}$. $\dim S = 2$

3. (20 points) Now consider the linear map $L : (P_2)^2 \to P_2$ given by $L(\vec{p}) = p_1 + p_2$. (Just add the two component polynomials.) For example, if $\vec{q} = \begin{pmatrix} -3+2x \\ 1+3x \end{pmatrix}$ then

$$L(\vec{q}) = L\begin{pmatrix} -3+2x \\ 1+3x \end{pmatrix} = (-3+2x) + (1+3x) = -2+5x$$

a. (5) Find the matrix of L relative to the e-basis on $(P_2)^2$ and the f-basis on P_2 where $f_1 = 1$ and $f_2 = x$. Call it $\underset{f\leftarrow e}{A}$.

$$\begin{aligned} L(e_1) &= L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 = f_1 \\ L(e_2) &= L\begin{pmatrix} x \\ 0 \end{pmatrix} = x = f_2 \\ L(e_3) &= L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 = f_1 \\ L(e_4) &= L\begin{pmatrix} 0 \\ x \end{pmatrix} = x = f_2 \end{aligned} \quad \Rightarrow \quad \underset{f\leftarrow e}{A} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

b. (5) Find the matrix of L relative to the E-basis on $(P_2)^2$ and the f-basis on P_2 by using the change of basis matrix. Call it $\underset{f\leftarrow E}{B}$.

$$\underset{f\leftarrow E}{B} = \underset{f\leftarrow e}{A} \; \underset{e\leftarrow E}{C} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

c. (5) Find the matrix of L relative to the E-basis on $(P_2)^2$ and the f-basis on P_2 from the definition.

$$\begin{aligned} L(E_1) &= L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 = f_1 \\ L(E_2) &= L\begin{pmatrix} 1+x \\ 0 \end{pmatrix} = 1+x = f_1 + f_2 \\ L(E_3) &= L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 = f_1 \\ L(E_4) &= L\begin{pmatrix} 0 \\ 1+x \end{pmatrix} = 1+x = f_1 + f_2 \end{aligned} \quad \Rightarrow \quad \underset{f\leftarrow E}{B} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

d. (5) If $(\vec{r})_E = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, what are $[L(\vec{r})]_f$ and $L(\vec{r})$?

$$[L(\vec{r})]_f = \underset{f\leftarrow E}{B} (\vec{r})_E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$$

$$L(\vec{r}) = 10f_1 + 6f_2 = 10 + 6x$$

4. (15 points) Again consider the linear map $L : (P_2)^2 \to P_2$ given by $L(\vec{p}) = p_1 + p_2$. When necessary, let $\vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} a + bx \\ c + dx \end{pmatrix}$.

a. (5) Find the kernel of L. Give a basis and the dimension.

$$Ker(L) = \{\vec{p} \mid L(\vec{p}) = p_1 + p_2 = 0\} = \{\vec{p} \mid p_2 = -p_1\} = \left\{\begin{pmatrix} a + bx \\ -a - bx \end{pmatrix}\right\} = \left\{a\begin{pmatrix} 1 \\ -1 \end{pmatrix} + b\begin{pmatrix} x \\ -x \end{pmatrix}\right\}$$

$$= Span\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} x \\ -x \end{pmatrix}\right\}$$

Basis: $\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} x \\ -x \end{pmatrix}\right\}$ $\quad \dim Ker(L) = 2$

b. (5) Find the image of L. Give a basis and the dimension.

$$\text{Im}(L) = \{L(\vec{p})\} = \{p_1 + p_2\} = \{a + bx + c + dx\} = \{(a + c)1 + (b + d)x\} = Span\{1, x\} = P_2$$

Basis: $\{1, x\}$ $\quad \dim \text{Im}(L) = 2$

c. (2) Is L one-to-one? Why?

L is NOT one-to-one because $Ker(L) \neq \{\vec{0}\}$.

d. (2) Is L onto? Why?

Yes, L is onto because $\text{Im}(L) = P_2 = Codom(L)$.

e. (1) Check that the Nullity-Rank Theorem is satisfied.

$$\dim Ker(L) + \dim \text{Im}(L) = 2 + 2 = 4 = \dim Dom(L)$$

5. (15 points) Verify that the following function is an inner product on $(P_2)^2$:

$$\langle\ ,\ \rangle : (P_2)^2 \times (P_2)^2 \to \mathbb{R} \quad \text{given by} \quad \langle \vec{p},\vec{q}\rangle = \int_{-1}^{1} p_1(x)q_1(x) + p_2(x)q_2(x)\,dx$$

For example, $\left\langle \begin{pmatrix} 1+x \\ 2x \end{pmatrix}, \begin{pmatrix} -x \\ 2-x \end{pmatrix} \right\rangle = \int_{-1}^{1} (1+x)(-x) + (2x)(2-x)\,dx = \int_{-1}^{1} (3x - 3x^2)\,dx = -2$

a. Symmetric:

$$\langle \vec{q},\vec{p}\rangle = \int_{-1}^{1} q_1(x)p_1(x) + q_2(x)p_2(x)\,dx = \int_{-1}^{1} p_1(x)q_1(x) + p_2(x)q_2(x)\,dx = \langle \vec{p},\vec{q}\rangle$$

b. Bilinear:

$$\begin{aligned}\langle \vec{p}, a\vec{q} + b\vec{r}\rangle &= \int_{-1}^{1} p_1(x)(aq_1 + br_1)(x) + p_2(x)(aq_2 + br_2)(x)\,dx \\ &= a\int_{-1}^{1} p_1(x)q_1(x) + p_2(x)q_2(x)\,dx + b\int_{-1}^{1} p_1(x)r_1(x) + p_2(x)r_2(x)\,dx \\ &= a\langle \vec{p},\vec{q}\rangle + b\langle \vec{p},\vec{r}\rangle\end{aligned}$$

c. Positive Definite:

$$\langle \vec{p},\vec{p}\rangle = \int_{-1}^{1} p_1(x)^2 + p_2(x)^2\,dx \geq 0$$

If $\langle \vec{p},\vec{p}\rangle = 0$, then $p_1(x) = p_2(x) = 0$. So $\vec{p} = \vec{0}$.

6. (15 points) Using the inner product of problem 5, find the angle between the vectors $\begin{pmatrix} 1 \\ x \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -x \end{pmatrix}$.

$$\left\langle \begin{pmatrix} 1 \\ x \end{pmatrix}, \begin{pmatrix} 1 \\ -x \end{pmatrix} \right\rangle = \int_{-1}^{1} (1)(1) + (x)(-x)\,dx = \int_{-1}^{1} (1 - x^2)\,dx = \frac{4}{3}$$

$$\left\langle \begin{pmatrix} 1 \\ x \end{pmatrix}, \begin{pmatrix} 1 \\ x \end{pmatrix} \right\rangle = \int_{-1}^{1} (1)(1) + (x)(x)\,dx = \int_{-1}^{1} (1 + x^2)\,dx = \frac{8}{3} \qquad \left|\begin{pmatrix} 1 \\ x \end{pmatrix}\right| = \sqrt{\frac{8}{3}}$$

$$\left\langle \begin{pmatrix} 1 \\ -x \end{pmatrix}, \begin{pmatrix} 1 \\ -x \end{pmatrix} \right\rangle = \int_{-1}^{1} (1)(1) + (-x)(-x)\,dx = \int_{-1}^{1} (1 + x^2)\,dx = \frac{8}{3} \qquad \left|\begin{pmatrix} 1 \\ -x \end{pmatrix}\right| = \sqrt{\frac{8}{3}}$$

$$\cos\theta = \frac{\left\langle \begin{pmatrix} 1 \\ x \end{pmatrix}, \begin{pmatrix} 1 \\ -x \end{pmatrix} \right\rangle}{\left|\begin{pmatrix} 1 \\ x \end{pmatrix}\right|\left|\begin{pmatrix} 1 \\ -x \end{pmatrix}\right|} = \frac{\frac{4}{3}}{\sqrt{\frac{8}{3}}\sqrt{\frac{8}{3}}} = \frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{\pi}{3} = 60^\circ$$