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MATH 311 Final Exam Spring 2003

Section 200 Solutions P. Yasskin

1	/25	3	/25
2	/25	4	/25

Work Out: (25 points each)

1. Consider an ideal gas whose density, ρ, temperature, T, and pressure, P, are functions of position. Thus if we consider a two dimensional space $\mathbb{R}^2$ whose coordinates are (ρ,T) then the ideal gas law, $P = k\rho T$, defines a function $P:\mathbb{R}^2 \to \mathbb{R}$. (Here k is a constant which may appear in your answers.) Further, the formulas which give (ρ,T) as functions of position (x,y,z) define a function $F:\mathbb{R}^3 \to \mathbb{R}^2$. The composition $P\circ F:\mathbb{R}^3 \to \mathbb{R}$ then gives P as a function of position. At the point $X=(2,3,4)$, ρ, T and their gradients are

$$\rho(X) = 2 \qquad T(X) = 78 \qquad \vec{\nabla}\rho(X) = (0.1, 0.2, -0.1) \qquad \vec{\nabla}T(X) = (0.2,-0.3,0.4)$$

a. What is $JF(X) = \dfrac{d(\rho,T)}{d(x,y,z)}(X)$, the Jacobian matrix of F at X?

$$JF = \begin{pmatrix} \frac{\partial\rho}{\partial x} & \frac{\partial\rho}{\partial y} & \frac{\partial\rho}{\partial z} \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{pmatrix} \qquad JF(X) = \begin{pmatrix} 0.1 & 0.2 & -0.1 \\ 0.2 & -0.3 & 0.4 \end{pmatrix}$$

b. What are JP and $JP(\rho(X),T(X))$, the Jacobian matrix of P and the Jacobian matrix of P at X?

$$JP = \begin{pmatrix} \frac{\partial P}{\partial\rho} & \frac{\partial P}{\partial T} \end{pmatrix} = \begin{pmatrix} kT & k\rho \end{pmatrix} \qquad JP(\rho(X),T(X)) = \begin{pmatrix} 78k & 2k \end{pmatrix}$$

c. What is $J(P\circ F)(X)$, the Jacobian matrix of $P\circ F$ at X?

$$J(P\circ F) = JP\,JF$$

$$J(P\circ F)(X) = JP(F(X))\,JF(X) = \begin{pmatrix} 78k & 2k \end{pmatrix}\begin{pmatrix} 0.1 & 0.2 & -0.1 \\ 0.2 & -0.3 & 0.4 \end{pmatrix} = \begin{pmatrix} 8.2k & 15k & -7k \end{pmatrix}$$

d. Use the linear approximation to estimate $P(Y)$, the pressure at the point $Y=(2.2, 2.9, 4.1)$.

$$P(X) = k\rho(X)T(X) = k\cdot 2\cdot 78 = 156k$$

$$\vec{\nabla}P = J(P\circ F)(X) = \begin{pmatrix} 8.2k & 15k & -7k \end{pmatrix}$$

$$Y-X = \begin{pmatrix} 2.2 \\ 2.9 \\ 4.1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0.2 \\ -0.1 \\ 0.1 \end{pmatrix}$$

$$P(Y) \approx P(X) + \vec{\nabla}P\cdot(Y-X) = 156k + \begin{pmatrix} 8.2k & 15k & -7k \end{pmatrix}\begin{pmatrix} 0.2 \\ -0.1 \\ 0.1 \end{pmatrix} = 155.44k$$

e. At the time $t=0$, you are at $X=(2,3,4)$ and moving with velocity $\vec{v}=(-1,1,2)$. Use the linear approximation to estimate the temperature T at time $t=2$.

$$T(2) = T(0) + \vec{\nabla}T(0)\cdot\vec{v}\cdot\Delta t = 78 + (0.2,-0.3,0.4)\cdot(-1,1,2)\,2 = 78.6$$

The remainder of the exam is customized for each student.