METRIC-CONNECTION
THEORIES OF GRAVITY

BY

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Dissertation submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1979

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Title of Thesis: Metric-Connection Theories of Gravity

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ABSTRACT

Title of Thesis: Metric-Connection Theories of Gravity

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In this thesis I study the metric-connection theories of gravity. These are defined as those theories in which the gravitational field is described by a metric and a connection which may be non-metric-compatible and/or non-torsion-free. In the broadest sense, this includes all of the metric theories. (These use the Christoffel connection.) However, I am mostly concerned with theories whose connection cannot be completely specified in terms of the metric. Although I sometimes consider completely general connections, I often restrict my attention to Cartan connections (metric-compatible but non-torsion-free).

In the context of metric-Cartan connection theories, I try to answer the question, is the torsion observable? Using a method similar to that used by Papapetrou and Dixon in the context of metric theories, I derive propagation equations for a body's momentum and angular momentum. These show that elementary particle spins feel the torsion but orbital angular momentum does not. However, the measurement of the effect of torsion on spin is beyond present technology. As a corollary, I prove that in a metric theory, spin and orbital angular momentum propagate in the same way.

In any metric-connection theory, the Christoffel connection is still present. I develop a two tangent space formalism which gives geometrical insight into the presence of two connections. The Christoffel connection
acts on one of the tangent spaces (called the external tangent space) while the full connection acts on the other tangent space (called the internal tangent space). The isomorphism between the two tangent spaces (called the soldering isomorphism) corresponds to the orthonormal frame in the usual one tangent space formalism. Its covariant derivative is the defect tensor which is the difference between the full connection and the Christoffel connection.

The two tangent space formalism suggests that the metric connection theories may be considered as the gauge theories of the spacetime symmetries. The role of the gauge potentials is played by the full connection and the soldering isomorphism. The spacetime symmetry group is the structure group of the internal tangent bundle and its specification may place restrictions on the full connection. The groups $O(3,1,R)$ and $SL(2,C)$ require a Cartan connection; $GL(4,R)$ allows a general connection; while $GL(2,C)$ unifies the electromagnetic potential with a Weyl-Cartan connection.

As an example of a computation using the two tangent space formalism, I rederive the conservation laws of energy-momentum and angular momentum by applying Noether's theorem to coordinate and $O(3,1,R)$-frame invariance. Considering $GL(4,R)$-frame invariance, I also obtain conservation laws for hypermomentum and dilation current.

The conservation laws and propagation equations only involve the kinematics of the gravitational fields. To understand their dynamics, one must choose gravitational field equations. I restrict my attention to field equations which involve no higher than second derivatives of the orthonormal frame and a Cartan connection. This is a different class of theories than the previously investigated metric-torsion theories which required no higher than second derivatives of the metric and torsion. The
Lagrangian can now be the Christoffel scalar curvature plus any function of the Cartan curvature and the torsion. This is a very large class of theories. I show that there is a twelve parameter family of such theories whose Lagrangians are quadratic polynomials in the Cartan curvature and torsion.

I further restrict my attention to the gravitational Lagrangian,

\[ L = \hat{R} + \hat{R}^\alpha_{\beta\gamma\delta} \hat{R}^\beta_{\gamma\alpha} , \]

where \( \hat{R}^\alpha_{\beta\gamma\delta} \) is the Cartan curvature. I verify that this theory has automatic Noether conservation laws and I prove a Birkhoff theorem which says that the unique 0(3)-spherically symmetric, vacuum solution is the Schwarzschild metric and zero torsion.
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In Memory of my Grandparents

Abraham and Mollie Rosenberg

and

Joseph and Lena Yasskin
ACKNOWLEDGMENTS

I am greatly indebted to many people for their help during my career at the University of Maryland. First of all, I thank my advisor, Robert Gowdy, whose pointed questions often led to a deeper insight into the physics of a problem. Special thanks go to Charles Misner, who suggested the investigation of the effects of torsion on the precession of a gyroscope; Paul Green, who patiently answered all my questions on fibre bundles; James Nester, who first introduced me to the notion of torsion; William Stoeger and Sriram Ramaswamy with whom I collaborated on portions of this research; and James Isenberg, whose insistence on an invariant formulation led me from the notion of mixed covariant derivatives to the two tangent space formalism.

I also thank Robert Wald, Frank Tipler, James Swank, Bahram Mashhoon, Lee Lindblom, Steve Lewis, William Hiscock, Jarmo Hietarinta, Oscar Greenberg, Mark Gotay, Steve Detweiler, John Dell, Dieter Brill, Edgar Beall and James Alexander for countless discussions on various topics in fibre bundles, gauge theories and general relativity.

I thank Joyce Alexander for typing the manuscript and putting up with my obsession with detail.

Portions of this research were supported by the Center for Theoretical Physics at the University of Maryland and by the NASA Grant NGR-21-002-010.

Finally, I wish to thank my parents and my girlfriend Sandy for their patience, understanding, encouragement and love.
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