## Sample problems for the final exam

Any problem may be altered or replaced by a different one!

**Problem 1 (15 pts.)** Find a quadratic polynomial p(x) such that p(-1) = p(3) = 6 and p'(2) = p(1).

**Problem 2 (20 pts.)** Consider a linear operator  $L : \mathbb{R}^3 \to \mathbb{R}^3$  given by

 $L(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{v}_1)\mathbf{v}_2$ , where  $\mathbf{v}_1 = (1, 2, -1)$ ,  $\mathbf{v}_2 = (2, 2, 1)$ .

- (i) Find the matrix of the operator L.
- (ii) Find the dimensions of the range and the kernel of L.
- (iii) Find bases for the range and the kernel of L.

**Problem 3 (20 pts.)** Let 
$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{pmatrix}$$
.

- (i) Evaluate the determinant of the matrix A.
- (ii) Find the inverse matrix  $A^{-1}$ .

**Problem 4 (25 pts.)** Let 
$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
.

- (i) Find all eigenvalues of the matrix B.
- (ii) Find a basis for  $\mathbb{R}^3$  consisting of eigenvectors of B.
- (iii) Find an orthonormal basis for  $\mathbb{R}^3$  consisting of eigenvectors of B.
- (iv) Find a diagonal matrix X and an invertible matrix U such that  $B = UXU^{-1}$ .

**Problem 5 (20 pts.)** Let V be a subspace of  $\mathbb{R}^4$  spanned by vectors  $\mathbf{x}_1 = (1, 1, 0, 0)$ ,  $\mathbf{x}_2 = (2, 0, -1, 1)$ , and  $\mathbf{x}_3 = (0, 1, 1, 0)$ .

(i) Find the distance from the point  $\mathbf{y} = (0, 0, 0, 4)$  to the subspace V.

(ii) Find the distance from the point **y** to the orthogonal complement  $V^{\perp}$ .

**Bonus Problem 6 (20 pts.)** (i) Find a matrix exponential  $\exp(tC)$ , where  $C = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$  and  $t \in \mathbb{R}$ .

(ii) Solve a system of differential equations

$$\begin{cases} \frac{dx}{dt} = 3x + y\\ \frac{dy}{dt} = 3y \end{cases}$$

subject to the initial conditions x(0) = y(0) = 1.

**Bonus Problem 7 (20 pts.)** Consider a linear operator  $K : \mathbb{R}^3 \to \mathbb{R}^3$  given by

$$K(\mathbf{x}) = D\mathbf{x}$$
, where  $D = \frac{1}{9} \begin{pmatrix} -4 & 7 & 4 \\ 1 & -4 & 8 \\ 8 & 4 & 1 \end{pmatrix}$ .

The operator K is a rotation about an axis.

- (i) Find the axis of rotation.
- (ii) Find the angle of rotation.