Math 304–504

Linear Algebra

Lecture 10a:
Cramer's rule.

A system of n linear equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Cramer's rule

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff A\mathbf{x} = \mathbf{b}$$

Theorem Assume that the matrix A is invertible. Then the only solution of the system is given by

$$x_i = \frac{\det A_i}{\det A}, \quad i = 1, 2, \ldots, n,$$

where the matrix A_i is obtained by substituting the vector $\mathbf{b}$ for the ith column of A.

Example.

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 13z = 1 \end{cases}$$

Augmented matrix of the system:

$$(A \,|\, \mathbf{b}) = \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 13 & 1 \end{array}\right)$$

As obtained in the previous lecture, $\det A = -12$. Since $\det A \neq 0$, there exists a unique solution of the system.

Example.

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 13z = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 13 & 1 \end{array}\right)$$

By Cramer's rule,

$$x = \frac{\begin{vmatrix} 0 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 8 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix}} = \frac{\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}}{-12} = \frac{-3}{-12} = \frac{1}{4}$$

Example.

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 13z = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 13 & 1 \end{array}\right)$$

By Cramer's rule,

$$y = \frac{\begin{vmatrix} 1 & 0 & 3 \\ 4 & 0 & 6 \\ 7 & 1 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix}} = \frac{-\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{-12} = \frac{6}{-12} = -\frac{1}{2}$$

Example.

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 13z = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 13 & 1 \end{array}\right)$$

By Cramer's rule,

$$z = \frac{\begin{vmatrix} 1 & 2 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}}{-12} = \frac{-3}{-12} = \frac{1}{4}$$

System of linear equations:

$$\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \\ 7x + 8y + 13z = 1 \end{cases}$$

Solution: $(x, y, z) = \left(\frac{1}{4}, -\frac{1}{2}, \frac{1}{4}\right).$