Math 304–504

Linear Algebra

Lecture 10b:
Vector spaces.

Linear operations on vectors

Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ be n-dimensional vectors, and $r \in \mathbb{R}$ be a scalar.

Vector sum: $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

Scalar multiple: $r\mathbf{x} = (rx_1, rx_2, \dots, rx_n)$

Zero vector: $\mathbf{0} = (0, 0, \dots, 0)$

Negative of a vector: $-\mathbf{y} = (-y_1, -y_2, \dots, -y_n)$

Vector difference:

$$\mathbf{x} - \mathbf{y} = \mathbf{x} + (-\mathbf{y}) = (x_1 - y_1, x_2 - y_2, \dots, x_n - y_n)$$

Properties of linear operations

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$
$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$$
$$\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$$
$$\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$$
$$r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$$
$$(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$$
$$(rs)\mathbf{x} = r(s\mathbf{x})$$
$$1\mathbf{x} = \mathbf{x}$$
$$0\mathbf{x} = \mathbf{0}$$

Linear operations on matrices

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m\times n$ matrices, and $r \in \mathbb{R}$ be a scalar.

Matrix sum: $A + B = (a_{ij} + b_{ij})_{1\le i\le m,\ 1\le j\le n}$

Scalar multiple: $rA = (ra_{ij})_{1\le i\le m,\ 1\le j\le n}$

Zero matrix O: all entries are zeros

Negative of a matrix: $-A = (-a_{ij})_{1\le i\le m,\ 1\le j\le n}$

Matrix difference: $A - B = (a_{ij} - b_{ij})_{1\le i\le m,\ 1\le j\le n}$

As far as the linear operations are concerned, the $m\times n$ matrices have the same properties as mn-dimensional vectors.

Vector space: informal description

Vector space = *linear space* = a set V of objects (called *vectors*) that can be added and scaled.

That is, for any $\mathbf{u}, \mathbf{v} \in V$ and $r \in \mathbb{R}$ expressions

$$\boxed{\mathbf{u} + \mathbf{v}} \text{ and } \boxed{r\mathbf{u}}$$

should make sense.

Certain restrictions apply. For instance,

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u},$$
$$2\mathbf{u} + 3\mathbf{u} = 5\mathbf{u}.$$

That is, addition and scalar multiplication in V should be like those of n-dimensional vectors.

Vector space: definition

Vector space is a set V equipped with two operations $\alpha : V \times V \to V$ and $\mu : \mathbb{R} \times V \to V$ that have certain properties (listed below).

The operation α is called *addition*. For any $\mathbf{u}, \mathbf{v} \in V$, the element $\alpha(\mathbf{u}, \mathbf{v})$ is denoted $\mathbf{u} + \mathbf{v}$.

The operation μ is called *scalar multiplication*. For any $r \in \mathbb{R}$ and $\mathbf{u} \in V$, the element $\mu(r, \mathbf{u})$ is denoted $r\mathbf{u}$.

Properties of addition and scalar multiplication (brief)

A1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

A2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$

A3. $\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$

A4. $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$

A5. $r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$

A6. $(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$

A7. $(rs)\mathbf{a} = r(s\mathbf{a})$

A8. $1\mathbf{a} = \mathbf{a}$

Properties of addition and scalar multiplication (detailed)

A1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ for all $\mathbf{a},\mathbf{b}\in V$.

A2. $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ for all $\mathbf{a},\mathbf{b},\mathbf{c}\in V$.

A3. There exists an element of V, called the *zero vector* and denoted $\mathbf{0}$, such that $\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}$ for all $\mathbf{a}\in V$.

A4. For any $\mathbf{a}\in V$ there exists an element of V, denoted $-\mathbf{a}$, such that $\mathbf{a}+(-\mathbf{a})=(-\mathbf{a})+\mathbf{a}=\mathbf{0}$.

A5. $r(\mathbf{a}+\mathbf{b})=r\mathbf{a}+r\mathbf{b}$ for all $r\in\mathbb{R}$ and $\mathbf{a},\mathbf{b}\in V$.

A6. $(r+s)\mathbf{a}=r\mathbf{a}+s\mathbf{a}$ for all $r,s\in\mathbb{R}$ and $\mathbf{a}\in V$.

A7. $(rs)\mathbf{a}=r(s\mathbf{a})$ for all $r,s\in\mathbb{R}$ and $\mathbf{a}\in V$.

A8. $1\mathbf{a}=\mathbf{a}$ for all $\mathbf{a}\in V$.

• Associativity of addition implies that a multiple sum $\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_k$ is well defined for any $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$.

• **Subtraction** in V is defined as usual: $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

• Addition and scalar multiplication are called **linear operations**.

Given $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$ and $r_1, r_2, \ldots, r_k \in \mathbb{R}$,

$$\boxed{r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_k\mathbf{u}_k}$$

is called a **linear combination** of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$.

Some general observations

- The zero vector is unique.

If $\mathbf{z}_1$ and $\mathbf{z}_2$ are zero vectors then $\mathbf{z}_1 = \mathbf{z}_1 + \mathbf{z}_2 = \mathbf{z}_2$.

- For any $\mathbf{a} \in V$, the negative $-\mathbf{a}$ is unique.

Suppose $\mathbf{b}$ and $\mathbf{b}'$ are negatives of $\mathbf{a}$. Then
$\mathbf{b}' = \mathbf{b}' + \mathbf{0} = \mathbf{b}' + (\mathbf{a} + \mathbf{b}) = (\mathbf{b}' + \mathbf{a}) + \mathbf{b} = \mathbf{0} + \mathbf{b} = \mathbf{b}$.

- $0\mathbf{a} = \mathbf{0}$ for any $\mathbf{a} \in V$.

Indeed, $0\mathbf{a} + \mathbf{a} = 0\mathbf{a} + 1\mathbf{a} = (0+1)\mathbf{a} = 1\mathbf{a} = \mathbf{a}$.
Then $0\mathbf{a} + \mathbf{a} = \mathbf{a} \implies 0\mathbf{a} = \mathbf{a} - \mathbf{a} = \mathbf{0}$.

- $(-1)\mathbf{a} = -\mathbf{a}$ for any $\mathbf{a} \in V$.

Indeed, $\mathbf{a} + (-1)\mathbf{a} = (-1)\mathbf{a} + \mathbf{a} = (-1)\mathbf{a} + 1\mathbf{a} = (-1+1)\mathbf{a}$
$= 0\mathbf{a} = \mathbf{0}$.

Examples of vector spaces

In most examples, addition and scalar multiplication are natural operations so that properties A1–A8 are easy to verify.

- $\mathbb{R}$: real numbers
- $\mathbb{Z}$: integers (**not** a linear space)
- $\mathbb{R}^n$ ($n \geq 1$): coordinate vectors
- $\mathbb{C}$: complex numbers
- $\mathcal{M}_{m,n}(\mathbb{R})$: $m \times n$ matrices with real entries
- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, \dots)$, $x_i \in \mathbb{R}$
- $\{0\}$: the trivial vector space