Math 304–504

Linear Algebra

Lecture 11:
Vector spaces and their subspaces.

Vector space

Vector space is a set V equipped with two operations $\alpha : V \times V \to V$ and $\mu : \mathbb{R} \times V \to V$ that have certain properties (listed below).

The operation α is called *addition*. For any $\mathbf{u}, \mathbf{v} \in V$, the element $\alpha(\mathbf{u}, \mathbf{v})$ is denoted $\mathbf{u} + \mathbf{v}$.

The operation μ is called *scalar multiplication*. For any $r \in \mathbb{R}$ and $\mathbf{u} \in V$, the element $\mu(r, \mathbf{u})$ is denoted $r\mathbf{u}$.

Properties of addition and scalar multiplication

A1. $\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ for all $\mathbf{a},\mathbf{b}\in V$.

A2. $(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ for all $\mathbf{a},\mathbf{b},\mathbf{c}\in V$.

A3. There exists an element of V, called the *zero vector* and denoted $\mathbf{0}$, such that $\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}$ for all $\mathbf{a}\in V$.

A4. For any $\mathbf{a}\in V$ there exists an element of V, denoted $-\mathbf{a}$, such that $\mathbf{a}+(-\mathbf{a})=(-\mathbf{a})+\mathbf{a}=\mathbf{0}$.

A5. $r(\mathbf{a}+\mathbf{b})=r\mathbf{a}+r\mathbf{b}$ for all $r\in\mathbb{R}$ and $\mathbf{a},\mathbf{b}\in V$.

A6. $(r+s)\mathbf{a}=r\mathbf{a}+s\mathbf{a}$ for all $r,s\in\mathbb{R}$ and $\mathbf{a}\in V$.

A7. $(rs)\mathbf{a}=r(s\mathbf{a})$ for all $r,s\in\mathbb{R}$ and $\mathbf{a}\in V$.

A8. $1\mathbf{a}=\mathbf{a}$ for all $\mathbf{a}\in V$.

• Associativity of addition implies that a multiple sum $\mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_k$ is well defined for any $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$.

• **Subtraction** in V is defined as usual: $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.

• Addition and scalar multiplication are called **linear operations**.

Given $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in V$ and $r_1, r_2, \ldots, r_k \in \mathbb{R}$,

$$\boxed{r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_k\mathbf{u}_k}$$

is called a **linear combination** of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$.

Additional properties of vector spaces

- The zero vector is unique.
- For any $\mathbf{a} \in V$, the negative $-\mathbf{a}$ is unique.
- $\mathbf{a} + \mathbf{b} = \mathbf{c} \iff \mathbf{a} = \mathbf{c} - \mathbf{b}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$.
- $\mathbf{a} + \mathbf{c} = \mathbf{b} + \mathbf{c} \iff \mathbf{a} = \mathbf{b}$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$.
- $0\mathbf{a} = \mathbf{0}$ for any $\mathbf{a} \in V$.
- $(-1)\mathbf{a} = -\mathbf{a}$ for any $\mathbf{a} \in V$.

Examples of vector spaces

In most examples, addition and scalar multiplication are natural operations so that properties A1–A8 are easy to verify.

- $\mathbb{R}$: real numbers
- $\mathbb{R}^n$ $(n \geq 1)$: coordinate vectors
- $\mathbb{C}$: complex numbers
- $\mathcal{M}_{m,n}(\mathbb{R})$: $m \times n$ matrices with real entries (also denoted $\mathbb{R}^{m \times n}$)
- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, \dots)$, $x_i \in \mathbb{R}$
- $\{0\}$: the trivial vector space

Functional vector spaces

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$
- $\widetilde{P}_n$: polynomials of degree n (**not** a vector space)
- $\mathcal{P}_n$: polynomials of degree at most n
- $F(\mathbb{R})$: all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$
- $F(\mathbb{R}) \setminus C(\mathbb{R})$: all discontinuous functions $f : \mathbb{R} \to \mathbb{R}$ (**not** a vector space)
- $C^1[a, b]$: all continuously differentiable functions $f : [a, b] \to \mathbb{R}$
- $C^\infty[a, b]$: all smooth functions $f : [a, b] \to \mathbb{R}$

Counterexample: dumb scaling

Consider the set $V = \mathbb{R}^n$ with the standard addition and a nonstandard scalar multiplication:

$$\boxed{r \odot \mathbf{a} = \mathbf{0}} \quad \text{for any } \mathbf{a} \in \mathbb{R}^n \text{ and } r \in \mathbb{R}.$$

Properties A1–A4 hold because they do not involve scalar multiplication.

A5. $r \odot (\mathbf{a} + \mathbf{b}) = r \odot \mathbf{a} + r \odot \mathbf{b} \iff \mathbf{0} = \mathbf{0} + \mathbf{0}$

A6. $(r + s) \odot \mathbf{a} = r \odot \mathbf{a} + s \odot \mathbf{a} \iff \mathbf{0} = \mathbf{0} + \mathbf{0}$

A7. $(rs) \odot \mathbf{a} = r \odot (s \odot \mathbf{a}) \iff \mathbf{0} = \mathbf{0}$

A8. $1 \odot \mathbf{a} = \mathbf{a} \iff \mathbf{0} = \mathbf{a}$

A8 is the only property that fails. As a consequence, property A8 does not follow from properties A1–A7.

Counterexample: lazy scaling

Consider the set $V = \mathbb{R}^n$ with the standard addition and a nonstandard scalar multiplication:

$$\boxed{r \odot \mathbf{a} = \mathbf{a}} \quad \text{for any } \mathbf{a} \in \mathbb{R}^n \text{ and } r \in \mathbb{R}.$$

Properties A1–A4 hold because they do not involve scalar multiplication.

A5. $r \odot (\mathbf{a} + \mathbf{b}) = r \odot \mathbf{a} + r \odot \mathbf{b} \iff \mathbf{a} + \mathbf{b} = \mathbf{a} + \mathbf{b}$

A6. $(r + s) \odot \mathbf{a} = r \odot \mathbf{a} + s \odot \mathbf{a} \iff \mathbf{a} = \mathbf{a} + \mathbf{a}$

A7. $(rs) \odot \mathbf{a} = r \odot (s \odot \mathbf{a}) \iff \mathbf{a} = \mathbf{a}$

A8. $1 \odot \mathbf{a} = \mathbf{a} \iff \mathbf{a} = \mathbf{a}$

The only property that fails is A6.

Subspaces of vector spaces

Definition. A vector space V_0 is a **subspace** of a vector space V if $V_0 \subset V$ and the linear operations on V_0 agree with the linear operations on V.

Examples.

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$
- $\mathcal{P}_n$: polynomials of degree at most n

$\mathcal{P}_n$ is a subspace of $\mathcal{P}$.

- $F(\mathbb{R})$: all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

$C(\mathbb{R})$ is a subspace of $F(\mathbb{R})$.

If S is a subset of a vector space V then S inherits from V addition and scalar multiplication. However S need not be closed under these operations.

Proposition A subset S of a vector space V is a subspace of V if and only if S is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$

$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{R}.$$

Proof: "only if" is obvious.
"if": properties like associative, commutative, or distributive law hold for S because they hold for V. We only need to verify properties A3 and A4. Take any $\mathbf{x} \in S$ (note that S is nonempty). Then $\mathbf{0} = 0\mathbf{x} \in S$. Also, $-\mathbf{x} = (-1)\mathbf{x} \in S$.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Any solution $(x_1, x_2, \ldots, x_n)$ is an element of $\mathbb{R}^n$.

Theorem The solution set of the system is a subspace of $\mathbb{R}^n$ if and only if all equations in the system are homogeneous (all $b_i = 0$).

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Proof: "only if": the zero vector $\mathbf{0} = (0, 0, \ldots, 0)$ is a solution only if all equations are homogeneous.

"if": if all equations are homogeneous then the solution set is not empty because it contains $\mathbf{0}$.

Suppose $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are solutions. That is, for every $1 \leq i \leq m$

$$
\begin{aligned}
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &= 0, \\
a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{in}y_n &= 0.
\end{aligned}
$$

Then $a_{i1}(x_1 + y_1) + a_{i2}(x_2 + y_2) + \cdots + a_{in}(x_n + y_n) = 0$ and $a_{i1}(rx_1) + a_{i2}(rx_2) + \cdots + a_{in}(rx_n) = 0$ for all $r \in \mathbb{R}$.

Hence $\mathbf{x} + \mathbf{y}$ and $r\mathbf{x}$ are also solutions.

Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$. Consider the set L of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{R}$.

Theorem L is a subspace of V.

Proof: First of all, L is not empty. For example, $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n$ belongs to L.

The set L is closed under addition since

$$(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n) + (s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n) =$$
$$= (r_1 + s_1)\mathbf{v}_1 + (r_2 + s_2)\mathbf{v}_2 + \cdots + (r_n + s_n)\mathbf{v}_n.$$

The set L is closed under scalar multiplication since

$$t(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n) = (tr_1)\mathbf{v}_1 + (tr_2)\mathbf{v}_2 + \cdots + (tr_n)\mathbf{v}_n.$$

Example. $V = \mathbb{R}^3$.

- The plane $z = 0$ is a subspace of $\mathbb{R}^3$.
- The plane $z = 1$ is not a subspace of $\mathbb{R}^3$.
- The line $t(1, 1, 0)$, $t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$ and a subspace of the plane $z = 0$.
- The line $(1, 1, 1) + t(1, -1, 0)$, $t \in \mathbb{R}$ is not a subspace of $\mathbb{R}^3$ as it lies in the plane $x + y + z = 3$, which does not contain $\mathbf{0}$.
- The plane $t_1(1, 0, 0) + t_2(0, 1, 1)$, $t_1, t_2 \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$.
- In general, a line or a plane in $\mathbb{R}^3$ is a subspace if and only if it passes through the origin.