Math 304–504

Linear Algebra

Lecture 12:
Span.

Subspaces of vector spaces

Definition. A vector space V_0 is a **subspace** of a vector space V if $V_0 \subset V$ and the linear operations on V_0 agree with the linear operations on V.

Examples.

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1x + \cdots + a_nx^n$
- $\mathcal{P}_n$: polynomials of degree **less than** n

$\mathcal{P}_n$ is a subspace of $\mathcal{P}$.

- $F(\mathbb{R})$: all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

$C(\mathbb{R})$ is a subspace of $F(\mathbb{R})$.

If S is a subset of a vector space V then S inherits from V addition and scalar multiplication. However S need not be closed under these operations.

Proposition A subset S of a vector space V is a subspace of V if and only if S is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$
$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{R}.$$

Remarks. The zero vector in a subspace is the same as the zero vector in V. Also, the subtraction in a subspace is the same as in V.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Any solution $(x_1, x_2, \ldots, x_n)$ is an element of $\mathbb{R}^n$.

Theorem The solution set of the system is a subspace of $\mathbb{R}^n$ if and only if all equations in the system are homogeneous (all $b_i = 0$).

Let V be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n \in V$. Consider the set L of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{R}$.

Theorem L is a subspace of V.

Definition. The subspace L is called the **span** of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ and denoted

$$\boxed{\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n).}$$

If $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n) = V$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is called a **spanning set** for V.

Remark. $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ is the minimal subspace of V that contains $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

Example. $V = \mathbb{R}^3$.

- The plane $z = 0$ is a subspace of $\mathbb{R}^3$.
- The line $t(1,1,0),\ t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$ and a subspace of the plane $z = 0$.
- The plane $t_1(1,0,0) + t_2(0,1,1),\ t_1, t_2 \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$.
- In general, a line or a plane in $\mathbb{R}^3$ is a subspace if and only if it passes through the origin.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$*:* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- diagonal matrices: $b = c = 0$
- upper triangular matrices: $c = 0$
- lower triangular matrices: $b = 0$
- symmetric matrices ($A^T = A$): $b = c$
- anti-symmetric matrices ($A^T = -A$):

$a = d = 0,\ c = -b$

- matrices with zero trace: $a + d = 0$

(trace = the sum of diagonal entries)

- matrices with zero determinant, $ad - bc = 0$,

do not form a subspace: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$:

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ consists of all matrices of the form

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

This is the subspace of diagonal matrices.

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ consists of all matrices of the form

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

This is the subspace of symmetric matrices.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$:

- The span of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the subspace of anti-symmetric matrices.

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the subspace of upper triangular matrices.

- The span of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is the entire space $\mathcal{M}_{2,2}(\mathbb{R})$.

Examples of subspaces of $C^\infty(\mathbb{R})$:

- $f''(x) - f(x) = 0, \ \ -\infty < x < \infty$

Solutions of this ODE form a subspace.

First of all, 0 is a solution. Further, if f and g are solutions, $f'' - f = 0$ and $g'' - g = 0$, then $(f+g)'' - (f+g) = 0$ and $(rf)'' - rf = 0$ for any $r \in \mathbb{R}$.

The subspace is spanned by functions e^x and e^{-x}.

- $f''(x) + f(x) = 0, \ \ -\infty < x < \infty$

Solutions form a subspace, which is spanned by functions $\sin x$ and $\cos x$.

- $f''(x) = 0, \ \ -\infty < x < \infty$

Solutions form a subspace, which is spanned by functions 1 and x.

Problem Let $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (3, 1, 1)$, and $\mathbf{w} = (4, -7, 3)$. Determine whether $\mathbf{w}$ belongs to $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

We have to check if there exist $r_1, r_2 \in \mathbb{R}$ such that $\mathbf{w} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2$. This vector equation is equivalent to a system of linear equations:

$$\begin{cases} 4 = r_1 + 3r_2 \\ -7 = 2r_1 + r_2 \\ 3 = 0r_1 + r_2 \end{cases} \iff \begin{cases} r_1 = -5 \\ r_2 = 3 \end{cases}$$

Thus $\mathbf{w} = -5\mathbf{v}_1 + 3\mathbf{v}_2 \in \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

Problem Let $\mathbf{v}_1 = (2,5)$ and $\mathbf{v}_2 = (1,3)$. Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a spanning set for $\mathbb{R}^2$.

Notice that $\mathbb{R}^2$ is spanned by vectors $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ since $(x,y) = x\mathbf{e}_1 + y\mathbf{e}_2$.

Hence it is enough to check that vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ belong to $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$. Then

$$\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) \supset \mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2) = \mathbb{R}^2.$$

$$\mathbf{e}_1 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 1 \\ 5r_1 + 3r_2 = 0 \end{cases} \iff \begin{cases} r_1 = 3 \\ r_2 = -5 \end{cases}$$

$$\mathbf{e}_2 = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 \iff \begin{cases} 2r_1 + r_2 = 0 \\ 5r_1 + 3r_2 = 1 \end{cases} \iff \begin{cases} r_1 = -1 \\ r_2 = 2 \end{cases}$$

Thus $\mathbf{e}_1 = 3\mathbf{v}_1 - 5\mathbf{v}_2$ and $\mathbf{e}_2 = -\mathbf{v}_1 + 2\mathbf{v}_2$.