Math 304–504

Linear Algebra

Lecture 13:
Linear independence.

Span: implicit definition

Let S be a subset of a vector space V.

Definition. The **span** of the set S, denoted $\mathrm{Span}(S)$, is the smallest subspace of V that contains S. That is,

- $\mathrm{Span}(S)$ is a subspace of V;
- for any subspace $W \subset V$ one has
$$S \subset W \implies \mathrm{Span}(S) \subset W.$$

Remark. The span of any set $S \subset V$ is well defined (it is the intersection of all subspaces of V that contain S).

Span: effective description

Let S be a subset of a vector space V.

- If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ then $\mathrm{Span}(S)$ is the set of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$, where $r_1, r_2, \dots, r_n \in \mathbb{R}$.

- If S is an infinite set then $\mathrm{Span}(S)$ is the set of all linear combinations $r_1\mathbf{u}_1 + r_2\mathbf{u}_2 + \cdots + r_k\mathbf{u}_k$, where $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k \in S$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$ $(k \geq 1)$.

- If S is the empty set then $\mathrm{Span}(S) = \{\mathbf{0}\}$.

Spanning set

Definition. A subset S of a vector space V is called a **spanning set** for V if $\mathrm{Span}(S) = V$.

Examples.

- Vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, and $\mathbf{e}_3 = (0, 0, 1)$ form a spanning set for $\mathbb{R}^3$ as

$$(x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$$

- Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ form a spanning set for $\mathcal{M}_{2,2}(\mathbb{R})$ as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

An infinite set $S \subset V$ is **linearly dependent** if there are some linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in S$. Otherwise S is **linearly independent**.

Theorem The following conditions are equivalent:
(i) vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly dependent;
(ii) one of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a linear combination of the other $k-1$ vectors.

Proof: (i) $\Longrightarrow$ (ii) Suppose that

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where $r_i \neq 0$ for some $1 \leq i \leq k$. Then

$$\mathbf{v}_i = -\tfrac{r_1}{r_i}\mathbf{v}_1 - \cdots - \tfrac{r_{i-1}}{r_i}\mathbf{v}_{i-1} - \tfrac{r_{i+1}}{r_i}\mathbf{v}_{i+1} - \cdots - \tfrac{r_k}{r_i}\mathbf{v}_k.$$

(ii) $\Longrightarrow$ (i) Suppose that

$$\mathbf{v}_i = s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k$$

for some scalars s_j. Then

$$s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k = \mathbf{0}.$$

Examples of linear independence

- Vectors $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, and $\mathbf{e}_3 = (0,0,1)$ in $\mathbb{R}^3$.

$$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = \mathbf{0} \implies (x,y,z) = \mathbf{0}$$
$$\implies x = y = z = 0$$

- Matrices $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

$$aE_{11} + bE_{12} + cE_{21} + dE_{22} = O \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} = O$$
$$\implies a = b = c = d = 0$$

Examples of linear independence

- Polynomials $1, x, x^2, \dots, x^n$.

$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0$ identically
$\implies \quad a_i = 0$ for $0 \le i \le n$

- The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$.

- Polynomials $p_1(x) = 1$, $p_2(x) = x - 1$, and $p_3(x) = (x-1)^2$.

$$a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = a_1 + a_2(x-1) + a_3(x-1)^2 =$$
$$= (a_1 - a_2 + a_3) + (a_2 - 2a_3)x + a_3 x^2.$$

Hence $a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) = 0$ identically
$\implies \quad a_1 - a_2 + a_3 = a_2 - 2a_3 = a_3 = 0$
$\implies \quad a_1 = a_2 = a_3 = 0$

Problem Let $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (3, 1, 1)$, and $\mathbf{v}_3 = (4, -7, 3)$. Determine whether vectors $\mathbf{v}_2, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

We have to check if there exist $r_1, r_2, r_3 \in \mathbb{R}$ not all zero such that $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3 = \mathbf{0}$.

This vector equation is equivalent to a system

$$\begin{cases} r_1 + 3r_2 + 4r_3 = 0 \\ 2r_1 + r_2 - 7r_3 = 0 \\ 0r_1 + r_2 + 3r_3 = 0 \end{cases} \qquad \left(\begin{array}{ccr|c} 1 & 3 & 4 & 0 \\ 2 & 1 & -7 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right)$$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent if and only if the matrix $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is singular. We obtain that $\det A = 0$.

Theorem Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n$ are linearly dependent whenever $m > n$.

Proof: Let $\mathbf{v}_j = (a_{1j}, a_{2j}, \dots, a_{nj})$ for $j = 1, 2, \dots, m$. Then the vector identity $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_m\mathbf{v}_m = \mathbf{0}$ is equivalent to the system

$$\begin{cases} a_{11}t_1 + a_{12}t_2 + \cdots + a_{1m}t_m = 0, \\ a_{21}t_1 + a_{22}t_2 + \cdots + a_{2m}t_m = 0, \\ \qquad\qquad \cdots\cdots\cdots \\ a_{n1}t_1 + a_{n2}t_2 + \cdots + a_{nm}t_m = 0. \end{cases}$$

Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are columns of the matrix (a_{ij}). If $m > n$ then the system is under-determined, therefore the zero solution is not unique.

Spanning sets and linear dependence

Let $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$ be vectors from a vector space V.

Proposition If $\mathbf{v}_0$ is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ then

$$\mathrm{Span}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k) = \mathrm{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k).$$

Indeed, if $\mathbf{v}_0 = r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k$, then

$$t_0\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k =$$
$$= (t_0r_1 + t_1)\mathbf{v}_1 + \cdots + (t_0r_k + t_k)\mathbf{v}_k.$$

Corollary Any spanning set for a vector space is minimal if and only if it is linearly independent.

Proposition Functions 1, e^x, and e^{-x} are linearly independent.

Proof: Suppose that $a + be^x + ce^{-x} = 0$ for some $a, b, c \in \mathbb{R}$. We have to show that $a = b = c = 0$.

$x = 0 \implies a + b + c = 0$
$x = 1 \implies a + be + ce^{-1} = 0$
$x = -1 \implies a + be^{-1} + ce = 0$

The matrix of the system is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & e & e^{-1} \\ 1 & e^{-1} & e \end{pmatrix}$.

$$\det A = e^2 - e^{-2} - 2e + 2e^{-1} =$$
$$= (e - e^{-1})(e + e^{-1}) - 2(e - e^{-1}) =$$
$$= (e - e^{-1})(e + e^{-1} - 2) = (e - e^{-1})(e^{1/2} - e^{-1/2})^2 \neq 0.$$

Hence the system has a unique solution $a = b = c = 0$.

Proposition Functions 1, e^x, and e^{-x} are linearly independent.

Alternative proof: Suppose that
$a + be^x + ce^{-x} = 0$ for some $a, b, c \in \mathbb{R}$.
Differentiate this identity twice:
$be^x - ce^{-x} = 0,$
$be^x + ce^{-x} = 0.$
It follows that $be^x = ce^{-x} = 0 \implies b = c = 0.$
Then $a = 0$ as well.

Theorem Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct real numbers. Then the functions $e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_k x}$ are linearly independent.

Furthermore, the set of functions $x^m e^{\lambda_i x}$, $1 \leq i \leq k$, $m = 0, 1, 2, \ldots$ is also linearly independent.